Model theory and analytic continuation

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1. Motivation

1.1 • Zilber's Conjecture: Every definable subset of $\mathbb{C}$ in the structure $\mathbb{C}_{exp} := \langle \mathbb{C}, +, \cdot, \exp \rangle$ is either countable or co-countable.

1.2 • Koiran's Conjecture: Every definable subset of $\mathbb{C}$ in the structure $\mathbb{C}_{1\text{-entire}} := \langle \mathbb{C}, +, \cdot, \{h\}_{h:\mathbb{C}\to\mathbb{C},\, h \text{ entire}} \rangle$ is either countable or co-countable.

~~Remark: A theorem of H. Alexander ('On a problem of Julia', Duke Math J. 42(2), 1975, 327–332) shows that if $F: \mathbb{C}^2 \to \mathbb{C}$ is entire, with non-empty irreducible zero set which is not of the form $\{\langle z, w\rangle \in \mathbb{C}^2 : z = z_0\}$.~~

1.3 Remark: Let $F: \mathbb{C}^2 \to \mathbb{C}$ be an entire function and assume that its zero set, Z_F, is non-empty, irreducible and not of the form $\{\langle z, w\rangle \in \mathbb{C}^2 : z = z_0\}$. Consider the projection, P_F, of Z_F onto the z-co-ordinate:–

$$P_F := \{z \in \mathbb{C} : \exists w \in \mathbb{C}\ F(z,w) = 0\}.$$

A theorem of Tsuji (Japanese J. Math (19) (1944), 139–154) asserts that $\mathbb{C} \setminus P_F$ has zero logarithmic capacity.
A theorem of H. Alexander (Duke Math J. (42) (1975), 327–332) asserts that for any such set $E \subseteq \mathbb{C}$, there exists $F: \mathbb{C}^2 \to \mathbb{C}$ as above such that $\mathbb{C} \setminus P_F = E$.
Since there exist uncountable sets of zero logarithmic capacity it follows that there exists $F: \mathbb{C}^2 \to \mathbb{C}$ with both P_F and $\mathbb{C} \setminus P_F$ uncountable, and hence the conjectures above fail in general for expansions of $\langle \mathbb{C}; +, \cdot \rangle$ by just one

entire function of two complex variables.

So even when considering the case of formulas $\exists w\ \tau(z,w)=0$, where τ is a term of, say, the Koiran language, we must use the special feature of τ that it is built up using polynomial functions from <u>unary</u> entire functions. I shall work out in detail a special case of when projections of zero sets of two variable terms of the Koiran language are co-countable. A corollary is:

1.4 <u>Theorem.</u>
Let $h:\mathbb{C}\to\mathbb{C}$ be an entire function of finite order and let $F(z,w) = \sum_{i,j,k=0}^{N} a_{i,j,k} z^i w^j h(w)^k$. Let $\langle z_0, w_0\rangle \in \mathbb{C}^2$ be such that $F(z_0,w_0) = 0 \neq \frac{\partial F}{\partial w}(z_0,w_0)$ and let $\phi: U\to\mathbb{C}$ be a function (as given by the implicit function theorem) satisfying $F(z,\phi(z)) = 0$ for all z in the open neighbourhood U of z_0. Then there exists (an explicitly specified) countable set $E\subseteq\mathbb{C}$ such that ϕ may be analytically continued along all rays $[z_0,\infty)$ emanating from z_0 satisfying $[z_0,\infty)\cap E = \emptyset$. Thus ϕ may be analytically continued along a piece-wise linear path to any point of $\mathbb{C}\setminus E$ and so $\mathbb{C}\setminus P_F \subseteq E$, and hence $\mathbb{C}\setminus P_F$ is countable.

2. <u>Asymptotic values.</u>
Let $h:\mathbb{C}\to\mathbb{C}\cup\{\infty\}$ be an meromorphic function. A complex number $\lambda\in\mathbb{C}$ is called an <u>asymptotic value</u> for h if there exists a smooth path $\gamma:[0,\infty)\to\mathbb{C}$ such that $|\gamma(t)|\to\infty$ and $h(\gamma(t))\to\lambda$ as $t\to\infty$. Let us call h <u>tame</u> if it has only countably many asymptotic values. (There are certainly examples of non-tame entire functions: the unit disk can even occur as the set of asymptotic values for some such h.)

• It is a theorem of Ahlfors (1930) that if h is entire and

$(*)\ldots\ \forall R>0,\ \sup_{|z|\le R}|h(z)| \le C\cdot e^{R^\rho}$ (for some fixed ρ ~~$\exists$~~, C ~~$\exists$~~ $\in \mathbb{R}$),

then h has at most 2ρ asymptotic values.

• Call h <u>very tame</u> if each ~~functions~~ function in the ring $\mathbb{C}[z, z^{-1}, h(z), h(z^2), \ldots]$ ~~has~~ has only countably many asymptotic values. It follows easily from Ahlfors theorem, that any entire function satisfying $(*)$ is <u>very tame</u>. (In fact each ~~functions~~ function in the above ring has only ~~countably~~ finitely many asymptotic values.) So 1.4 follows from:–

2.1 <u>Theorem.</u>
Theorem 1.4 holds for very tame functions h.

3. <u>Proof of 2.1</u>
We first specify the set E. (We use the notations of the statement of 1.4.)

Firstly, it is clear that on the connected component of $Z(F)$ $(:= \{\langle z,w\rangle \in \mathbb{C}^2 : F(z,w)=0\})$ containing $\langle z_0, w_0\rangle$, the set $\{\langle z,w\rangle \in \mathbb{C}^2 : \frac{\partial F}{\partial w}(z,w)=0\}$ is discrete. Let E_{sing} be its projection onto z-coordinates: E_{sing} is countable.

<u>3.1</u> • It clearly follows that if $[z_0, z_1)$ is a ray emanating from z_0 along which ϕ can be analytically continued, and if $[z_0,z_1)\cap E_{sing} = \emptyset$, then ϕ can be continued to $[z_0, z_1]$ <u>provided</u> it is not the case that $\lim_{\substack{z\to z_1\\ z\in[z_0,z_1)}} \phi(z) = \infty$.

(For otherwise, there would exist a sequence $\{u_n\} \subseteq [z_0, z_1)$ s.th. $u_n \to z_1$ and $\phi(u_n) \to$ some $\theta \in \mathbb{C}$, as $n\to\infty$. We would then have $F(u_n, \phi(u_n)) = 0$ $(\forall n)$, and hence $F(z_1,\theta)=0$ and since $\frac{\partial F}{\partial z}(z_1,\theta)\neq 0$ (as $z_1 \notin E_{sing}$) we could continue ϕ by the Implicit

Function Theorem.)

- We now construct subfields $E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots$ of $\mathbb{C}$ inductively as follows.
  - Let $E_0 = \mathbb{Q}(\{a_{i,j,k} : 0 \leq i,j,k \leq m\})$ (see defn. of F in 1.4);
  - Let E_{n+1} = subfield of $\mathbb{C}$ generated over E_n by all asymptotic values of all functions in the ring $E_n[z, z^{-1}, h(z), h(z^2), \ldots]$
  - Let $E_\infty = \bigcup_{n \geq 0} E_n$, and define K to be the algebraic closure of $\mathbb{Q}(E_\infty \cup E_{sing})$ (within $\mathbb{C}$).
- Let $R :=$ the ring $K[z, w, h(w)]$ (considered as a subring of the ring of all entire functions of two complex variables). Clearly $F(z,w) \in R$ and both K and R are countable by our assumption of h being very tame.

We now define the exceptional set. Firstly, let

3.2 $\tilde{E} := \{z \in \mathbb{C} : \exists w \in \mathbb{C}, \exists G \in R$ such that $\langle z, w\rangle$ is an isolated zero of G on the connected component of $Z(F)$ containing $\langle z_0, w_0\rangle\}$

3.3 $E :=$ the algebraic closure of $K(\tilde{E})$ (within $\mathbb{C}$).

- Since R is countable, so is E. We complete the proof of 2.1 by (in the notation of 3.1) deriving a contradiction from the two assumptions:

  3.4 $\phi(z) \to \infty$ as $z \to z_1$ along $[z_0, z_1)$;

  3.5 $[z_0, z_1] \cap E = \emptyset$.

(Note that E does not depend on the point z_1! It does depend on the initial point $\langle z_0, w_0\rangle$ (as we refer to the connected component of $Z(F)$ containing $\langle z_0, w_0\rangle$), but this could easily be avoided. Thus E really only depends on h and the coefficients of F.)

- So, assume 3.4 and 3.5.

Let R^ϕ denote the ring $K[z, \phi(z), h(\phi(z))]$, considered as a subring of the ring of analytic functions from $[z_0, z_1)$ to $\mathbb{C}$.

Notice that if $f \in R^\phi$, $f \not\equiv 0$, then f has no zeroes (since if z' were such a zero, then, letting $G \in R$ be such that $f(z) \equiv G(z, \phi(z))$, we would have $\langle z', \phi(z')\rangle$ being an isolated zero of G on the connected component of $Z(F)$ containing $\langle z_0, w_0\rangle$, and hence $z' \in \tilde{E} \cap [z_0, z_1)$ (by 3.2), so $z' \in [z_0, z_1] \cap E$ – contradicting 3.5).

Thus, the field of fractions $\mathcal{F}^\phi := K(z, \phi(z), h(\phi(z)))$ of R^ϕ is also a subring of the ring of analytic functions from $[z_0, z_1)$ to $\mathbb{C}$.

3.6 <u>Claim</u>: For any $f \in \mathcal{F}^\phi$, $\displaystyle\lim_{\substack{z \to z_1 \\ z \in [z_0, z_1)}} f(z)$ exists in $\mathbb{C} \cup \{\infty\}$.

<u>Proof of claim</u>:

~~For~~ suppose not for some $f \in \mathcal{F}^\phi$. Since $f(z) \not\to \infty$ (as $z \to z_1$), there exists a sequence $\{s_n\} \subseteq [z_0, z_1)$ and $\theta \in \mathbb{C}$ such that $s_n \to z_1$ and $f(s_n) \to \theta$ as $n \to \infty$. Since, also, $f(z) \not\to \theta$ as $z \to z_1$, there exists a sequence $\{s_n'\} \subseteq [z_0, z_1)$ and $\theta' \in \mathbb{C}\setminus\{\theta\}$ such that $s_n' \to z_1$ and $f(s_n') \to \theta'$ as $n \to \infty$. But since $f : [z_0, z_1) \to \mathbb{C}$ is continuous, it easily follows that $|f(z) - \theta|$ takes all values in $(0, |\theta - \theta'|)$ arbitrarily close to z_1. Thus there exist uncountably many points $\alpha \in \mathbb{C}$ for which $f(s_n^\alpha) \to \alpha$, for some sequence $\{s_n^\alpha\} \subseteq [z_0, z_1)$ with $s_n^\alpha \to z_1$. In particular we may choose $\alpha \in \mathbb{C}$ transcendental over $K(z_1)$ with this property, which we now do.

Now pass to a proper ultrapower ${}^*\mathbb{C}$ of $\mathbb{C}$ and let ω be an infinite natural number. Let $\lambda := s_\omega^\alpha$. Then $st(\lambda) = z_1$ and $st(f(\lambda)) = \alpha$ (where $st(\cdot)$ denotes taking the standard part). Since α is transcendental over z_1 and $K \subseteq \mathbb{C}$, it follows that the subfield $K(\lambda, f(\lambda))$ of ${}^*\mathbb{C}$ is isomorphic to $K(\lambda, \alpha)$ via the standard part map, and hence that $K(\lambda, f(\lambda))$ is a subfield of the ring $\mathrm{Fin}({}^*\mathbb{C})$ ($:=$ finite elements of ${}^*\mathbb{C}$). However, by 3.4, $\phi(\lambda)$ ($= \phi(s_\omega^\alpha)$ evaluated in ${}^*\mathbb{C}$) is an infinite element of ${}^*\mathbb{C}$ so clearly the subfield $K(\lambda, f(\lambda), \phi(\lambda))$ of ${}^*\mathbb{C}$ has transcendence degree 3 over K. But this subfield is certainly contained in the subfield $K(\lambda, \phi(\lambda), h(\phi(\lambda)))$ of ${}^*\mathbb{C}$ which has (assuming our original function F is not identically zero) transcendence degree 2 over K – contradiction. $\square$ 3.6.

Now 3.6 allows us to view $\mathcal{F}^\phi$ as a valued field. The valuation ring is $V := \{ f \in \mathcal{F}^\phi : \lim_{\substack{z \to z_1 \\ z \in [z_0, z_1)}} f(z) \text{ is finite} \}$,

and its maximal ideal is $\mu := \{ f \in \mathcal{F}^\phi : \lim_{\substack{z \to z_1 \\ z \in [z_0, z_1)}} f(z) = 0 \}$.

Notice that the residue field, k say, contains (up to isomorphism) $K(z_1)$ (NB $\lim_{\substack{z \to z_1 \\ z \in [z_0, z_1)}} z = z_1$), which has transcendence degree 1 over K, and the value group is non-trivial (~~xxxx~~ by 3.4). Since $\mathcal{F}^\phi$ has transcendence degree 2 over K, it follows that k is the algebraic closure of $K(z_1)$ (i.e. of $K(z)$) in $\mathcal{F}^\phi$, and the value group ~~has~~ the form $\{ \frac{m}{q} \cdot v(\phi(z)) : m \in \mathbb{Z} \}$, where $q \in \mathbb{Z}, q > 0$ is fixed, and v denotes the valuation.

Now since we may assume that $h(\phi(z))$ is algebraic over $K(z, \phi(z))$ (otherwise $F(z, w)$ is a polynomial, and then the main theorem is trivial) it follows that we have a formal Puiseux series expansion (at infinity):

$$3.7 \qquad h(\phi(z)) = \sum_{i=-l}^{\infty} a_i \cdot \phi(z)^{-i/q},$$

where the a_i's lie in the residue field k ($:=$ alg. clos. $K(z_1)$).

This expression 3.7 is to be read, of course, in terms of the valuation, i.e.

- $h(\phi(z)) \cdot \phi(z)^{-l/q} \to a_{-l}$ as $z \to z_1$ along $[z_0, z_1)$,
- $\phi(z)^{-\frac{l}{q}}\left(h(\phi(z)) \cdot \phi(z)^{-l/q} - a_{-l}\right) \to a_{-l+1}$ — " —
- $\phi(z)^{1/q}\left(\phi(z)^{1/q}\left(h(\phi(z)) \cdot \phi(z)^{-l/q} - a_{-l}\right) - a_{-l+1}\right) \to a_{-l+2}$ — " —

$\vdots$

etc.

So we see that (by 3.4),

- a_{-l} is an asymptotic value for the function
$$h(w^q) \cdot w^{-l} \quad ;$$
- a_{-l+1} is an asymptotic value for the function
$$w\left(h(w^q) \cdot w^{-l} - a_{-l}\right) \quad ;$$
- a_{-l+2} is an asymptotic value for the function
$$w\left(w\left(h(w^q) \cdot w^{-l} - a_{-l}\right) - a_{-l+1}\right) \quad ;$$

and so on.

Thus, it follows inductively that $a_{-l}, a_{-l+1}, a_{-l+2}, \ldots$ all lie in E_∞, and hence in K.

So, in fact, the series in 3.7 is in fact a series with coefficients in K and it follows from this that $h(\phi(z))$ is algebraic over $K(\phi(z))$ (and not just algebraic over $K(z, \phi(z))$). So there exists a $Q(X,Y) \in K[X,Y]$ such that $Q(\phi(z), h(\phi(z))) = 0$ (in the ring of analytic functions from $[z_0, z_1)$ to $\mathbb{C}$). But then, by analyticity, $Q(w, h(w)) \equiv 0$, and h is an algebraic function, and hence so is ϕ, and the main result is well-known in this case. $\square$