Zariski structures and noncommutative geometry

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http://www.people.maths.ox.ac.uk/ ∼zilber:
Zariki Geometries (forthcoming book);
A class of quantum Zariski geometries;
Non-commutative Zariski geometries and their classical limit;
Quantum Harmonic Oscillator as a Zariski Geometry.

Zariski structures

Zariski structures (1993, E.Hrushovski and B.Zilber) are on the very top of the (logical) stability hierarchy. The ones for which a fine classification theory is possible.

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(T) Zariski closed sets form a Noetherian Topology on M^n, all n.

(P) Projection $\mathrm{pr}(S) \subseteq M^n$ of a closed set $S \subseteq M^{n+1}$ is constructible (= Boolean combination of closed).

(D) Dimension $\dim S$ to any closed $S \subseteq M^n$ is assigned.

(AF) Addition formula:

$$\dim S = \dim \mathrm{pr}(S) + \min_{a \in pr(S)} \dim(\mathrm{pr}^{-1}(a) \cap S)$$

for any closed irreducible S.

(FC) Fiber condition: for each k, the set

$$\{a \in M^{n-1} : \dim(S \cap pr^{-1}(a)) > k\}$$

is constructible.

(PS) Pre-smoothness: For any closed irreducible $S_1, S_2 \subseteq M^n$,

$$\dim S_1 \cap S_2 \geq \dim S_1 + \dim S_2 - \dim M^n$$

in each component.

Known classes of Zariski structures

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4. *A large class of non-commutative geometries* (2005)

About the term *Geometry*.

Geometric tradition explains "spaces" as given locally by co-ordinate functions (into $\mathbb{R}$ or $\mathbb{C}$). This follows the physicist's paradigm that the ultimate data is given in numbers.

Classification Theorem (Hrushovski, Zilber 1993)
For any non-linear Zariski geometry M there is an algebraically closed field $\mathbb{F}$ and a nonconstant *meromorphic* function

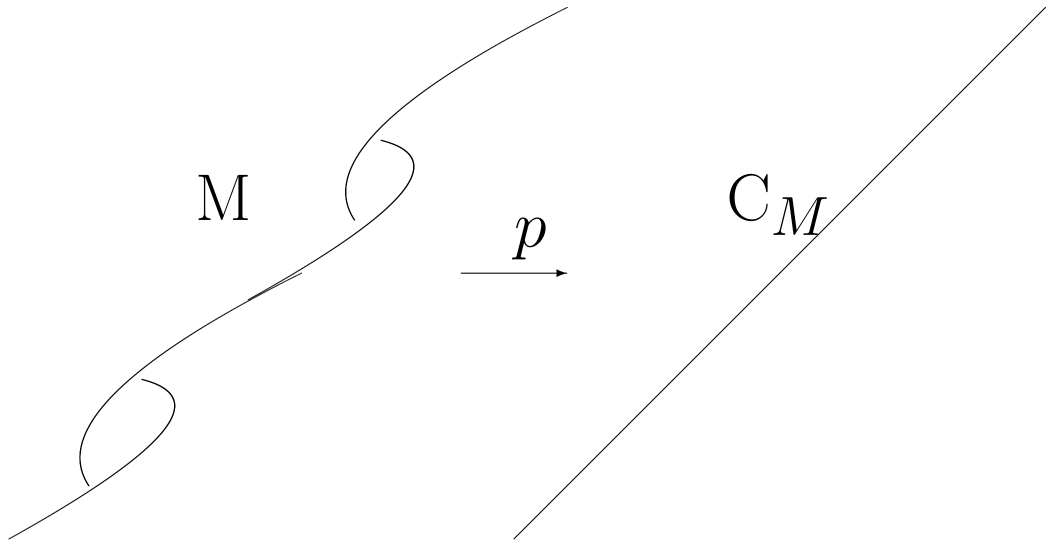
$$f : \mathrm{M} \to \mathbb{F}.$$

In particular, if $\dim M = 1$ then there is a smooth projective algebraic curve C_{M} and a Zariski-continuou finite covering map

$$p : \mathrm{M} \to C_{\mathrm{M}}(\mathbb{F}),$$

the image of any relation on M is just an algebraic relation on C_{M}.

Zariski structure and its algebraic projection



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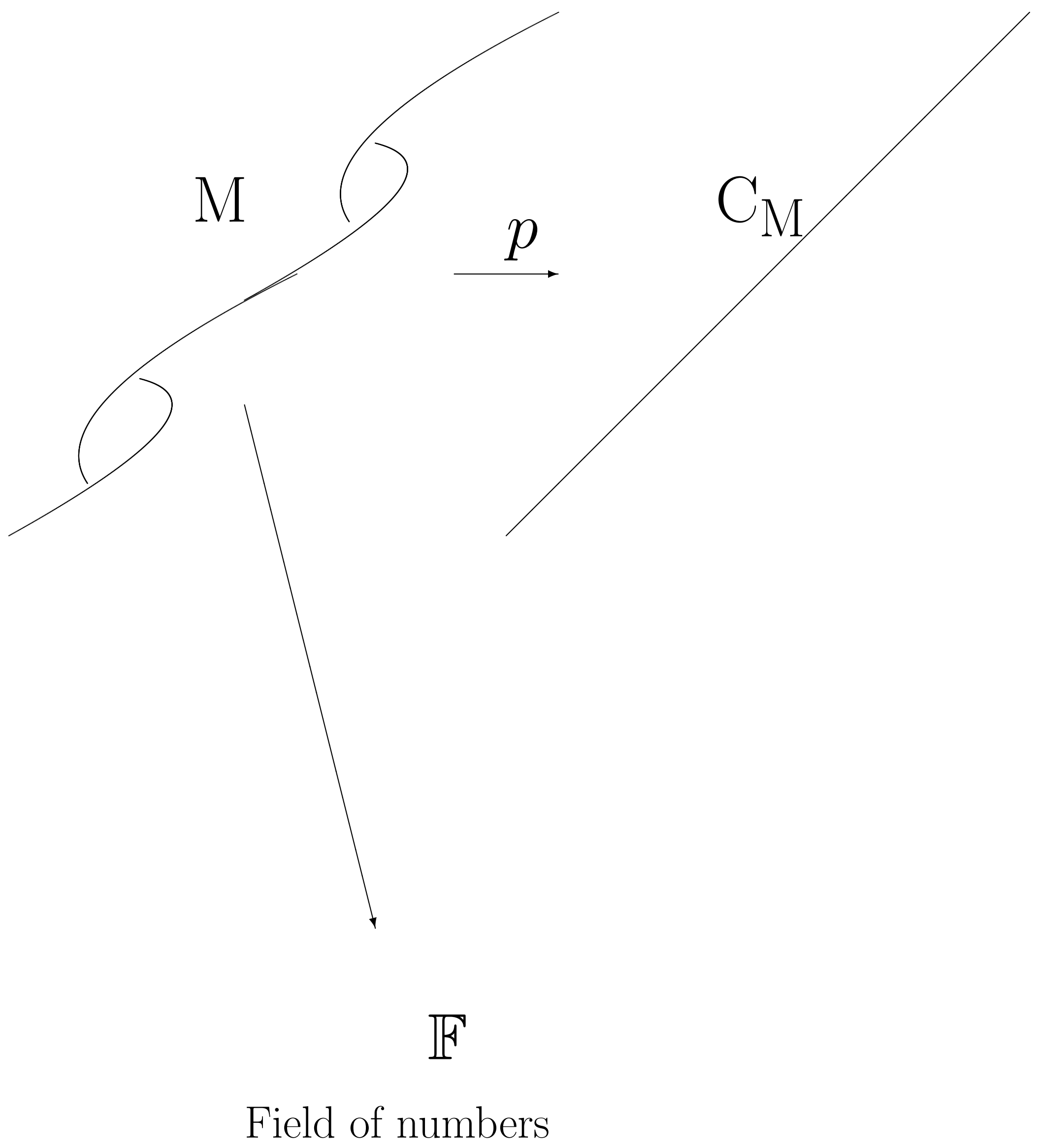

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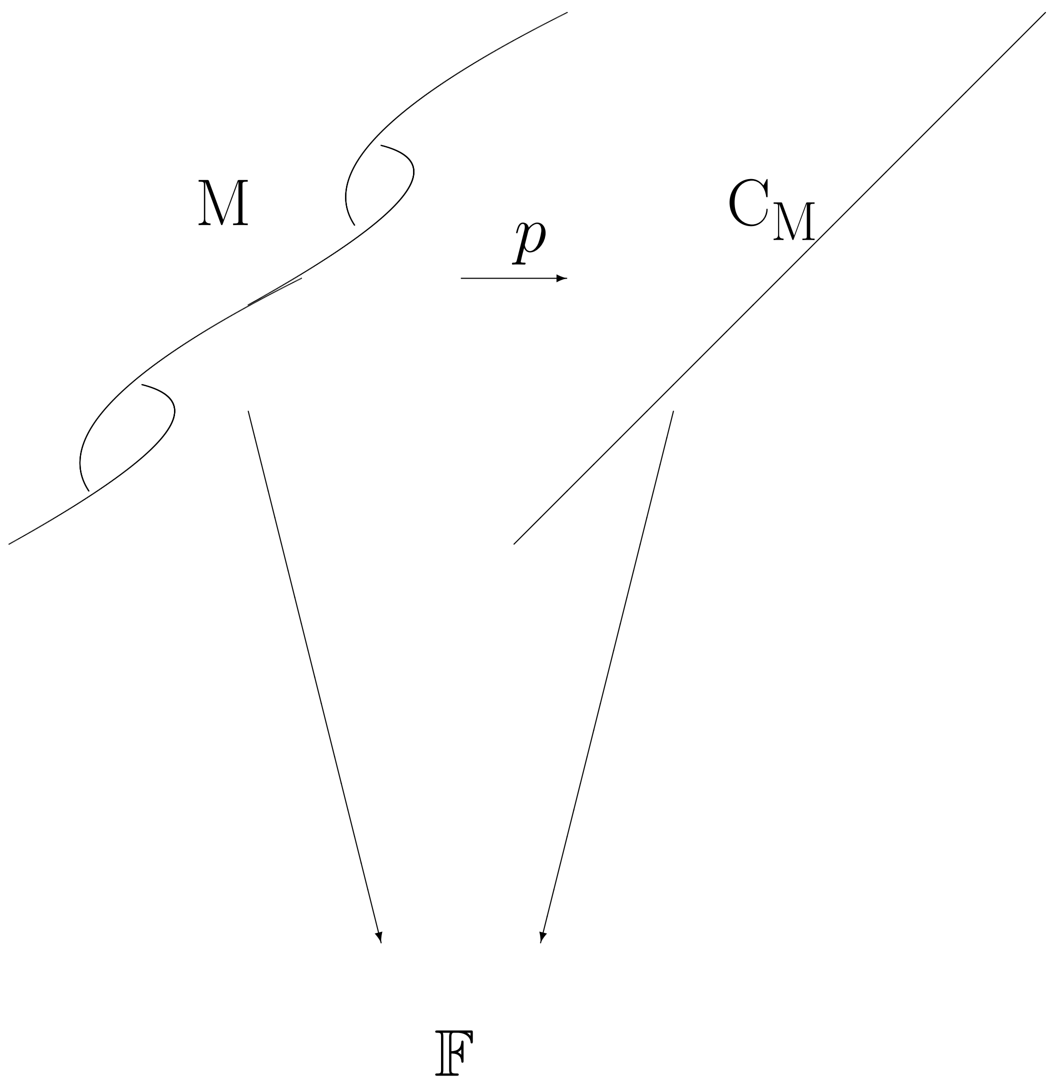

Field of numbers

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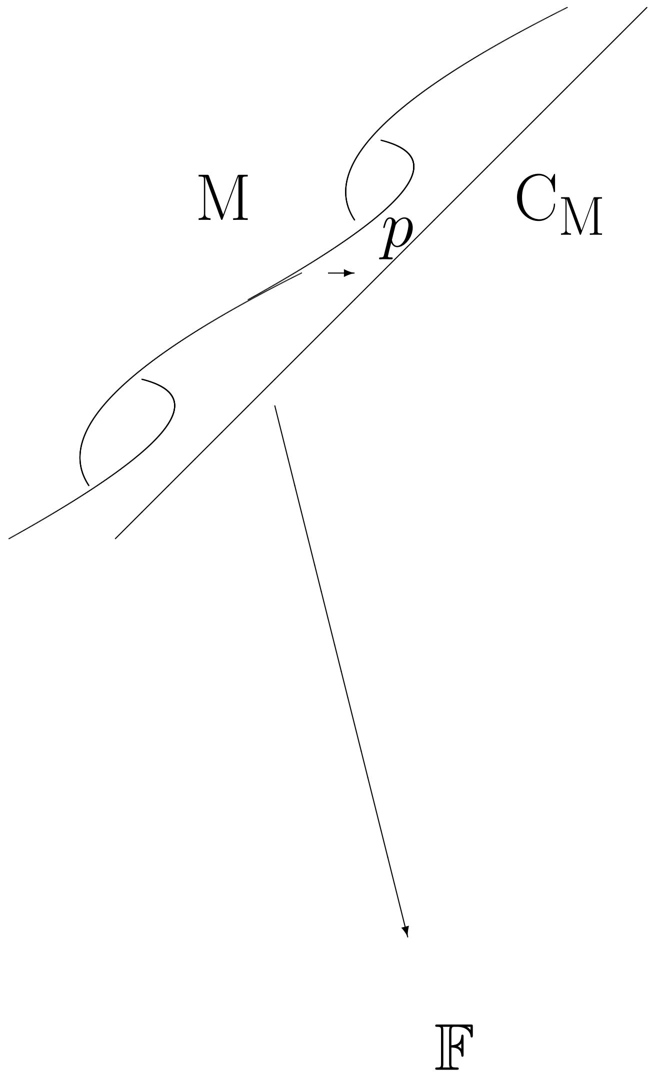

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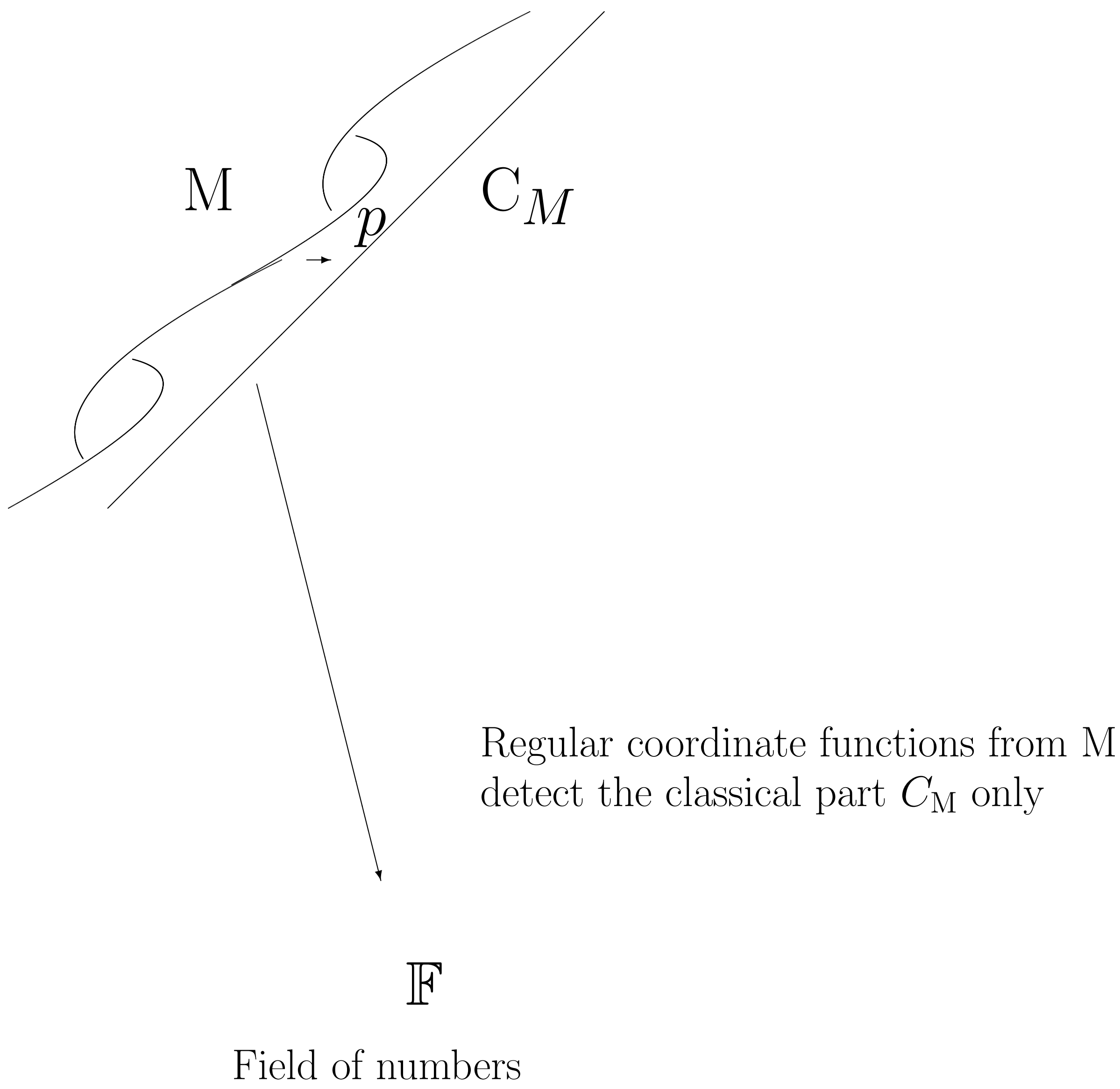


$$\mathbb{F}[\mathrm{M}] = \{f : \mathrm{M} \to \mathbb{F} \text{ regular } \}$$

$$\mathbb{F}[\mathrm{M}] = \mathbb{F}(C_{\mathrm{M}}), \quad C_{\mathrm{M}} = M/E,$$

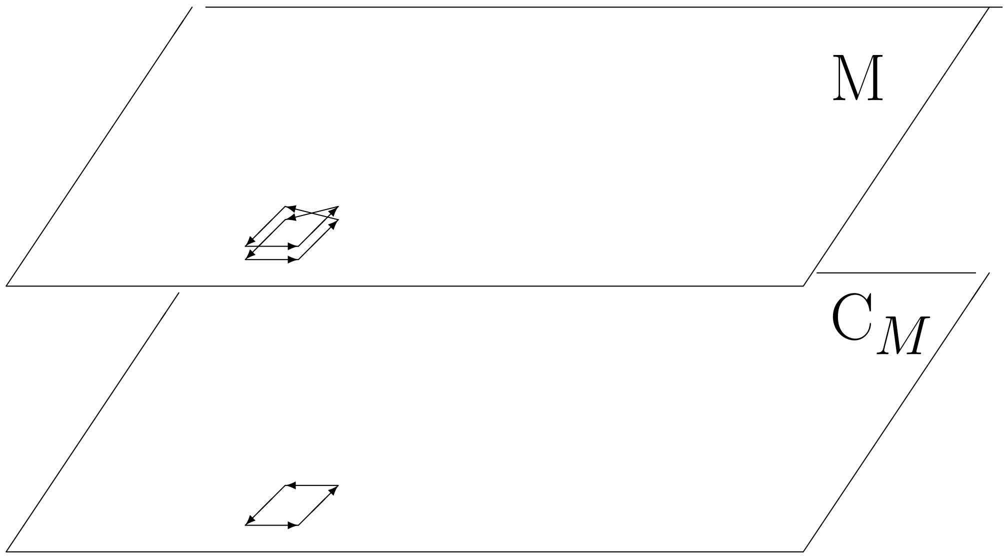
E an equivalence relation on M.

In general there may be Zariski-continuous "entangling" arrows (action)

$$\gamma : \ \mathrm{M} \to \mathrm{M}, \qquad \gamma \in \Gamma$$

which make E non-splitting.

The initial example: 2-cover of the affine line.



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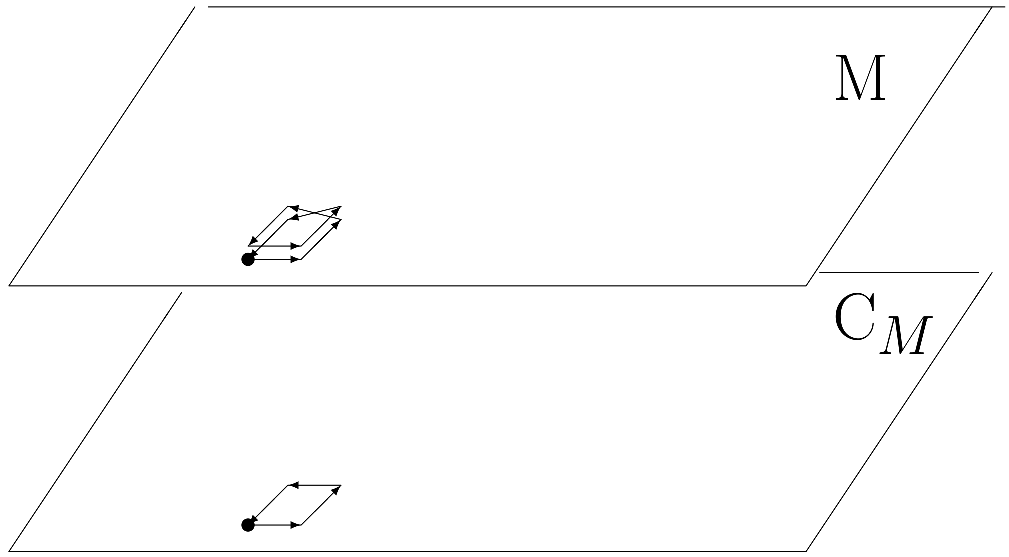

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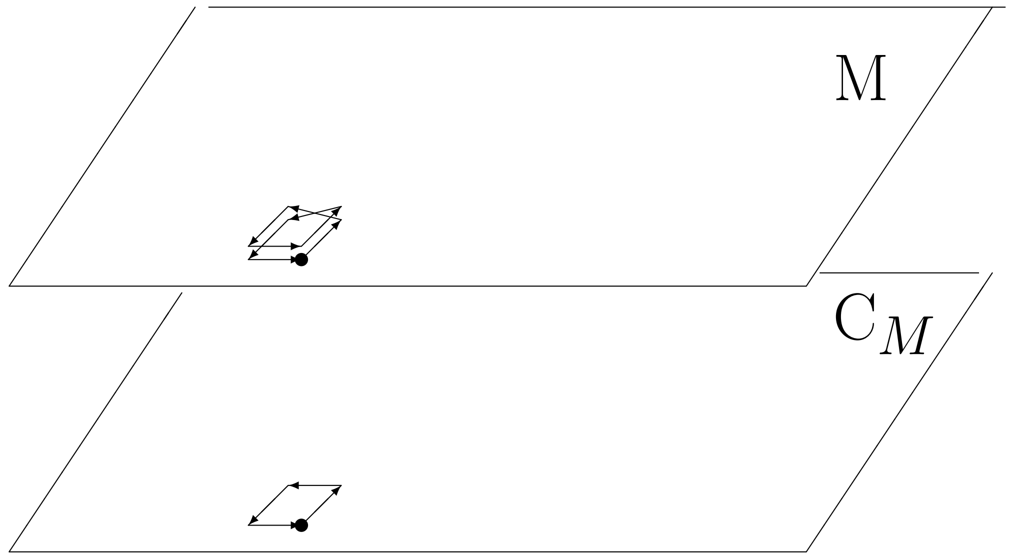

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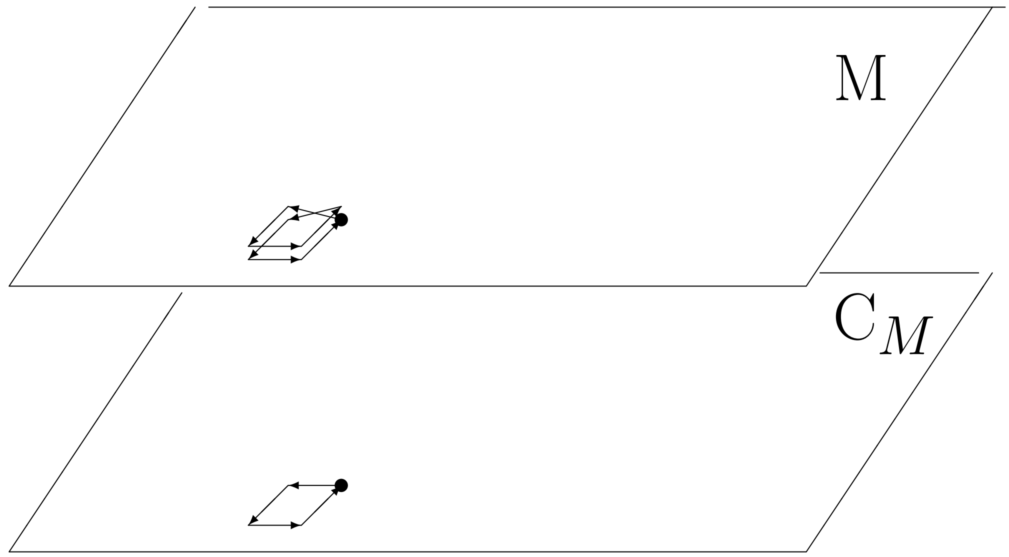

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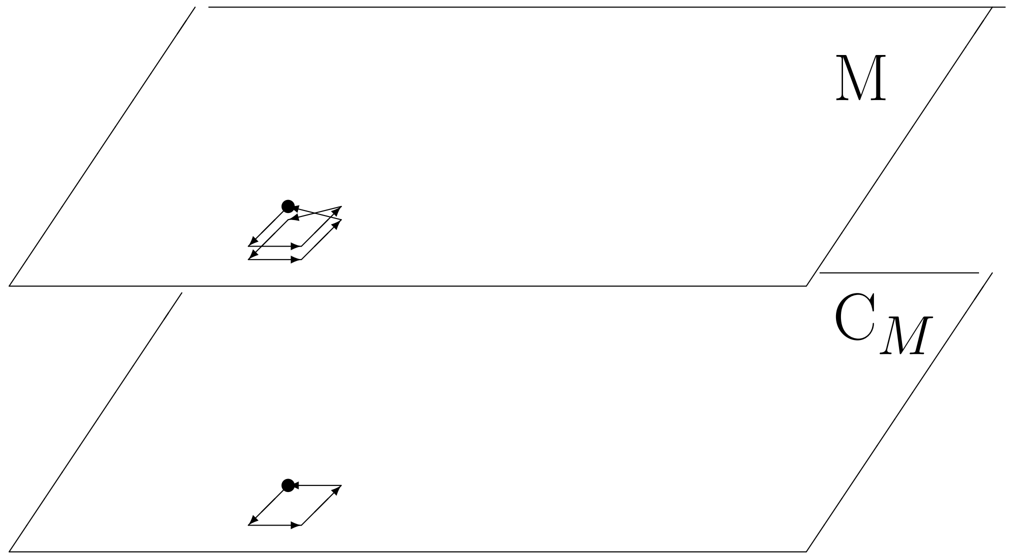

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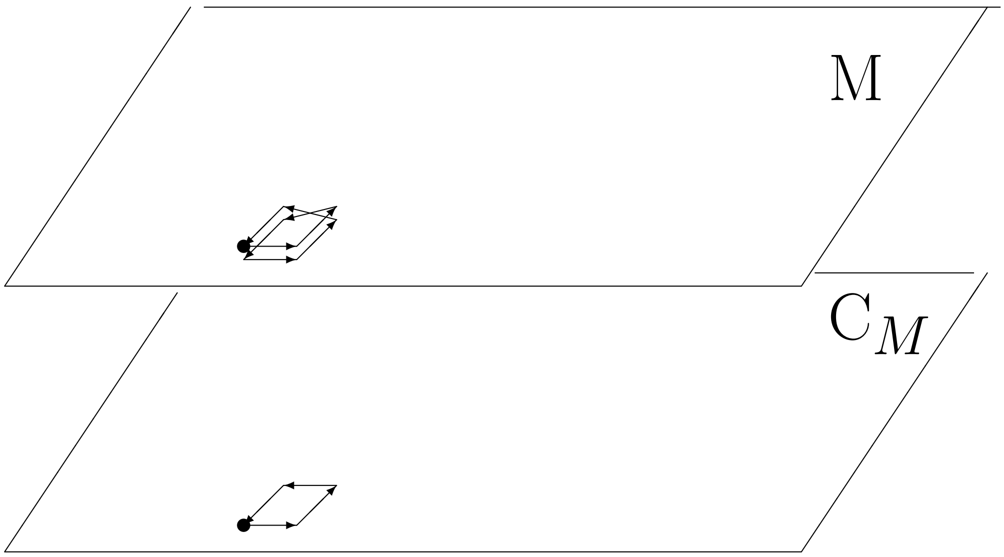

Classification theorem revisited.

How to recover the hidden relations in terms of “coordinate functions”?

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Extend the $\mathbb{F}$-algebra of definable functions $\mathbb{F}[\mathrm{M}]$ to the $\mathbb{F}$-space of semi-definable functions $\mathcal{H}[\mathrm{M}]$.

Every Zariski bijection γ generates an $\mathbb{F}$-linear transformation of $\mathcal{H}[\mathrm{M}]$:

$$U_\gamma : \ f \mapsto f^\gamma$$

$$f \in \mathcal{H}[\mathrm{M}], \quad f^\gamma(x) = f(\gamma x).$$

Also, any $y \in \mathbb{F}[\mathrm{M}]$ gives rise to an $\mathbb{F}$-linear

$$Y : f \mapsto y \cdot f$$

Classification theorem revisited.

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The operator algebra $A[\mathrm{M}]$ *generated by all the* U_γ *and* Y*'s contains data sufficient to recover* M.

$$M \longrightarrow \mathbb{F}[M]$$

$$\mathrm{M} \longrightarrow \mathbb{F}[\mathrm{M}] \subset \mathcal{H}[\mathrm{M}]$$

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Remarks

1. Elements of $\mathcal{H}[M]$ are not uniquely definable within M, so should be considered as auxiliary, not well-defined.

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Remarks

1. Elements of $\mathcal{H}[\mathrm{M}]$ are not uniquely definable within M, so should be considered as auxiliary, not well-defined.

2. $A[\mathrm{M}]$ and its elements are uniquely defined (up to the choice of the language) so can be seen as **observables**.

$$\left\{ \begin{array}{l} \text{universe} \\ \text{of M} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{class of 1-dim rep of} \\ \text{a commutative } B \leq A[\mathrm{M}] \end{array} \right\}$$

Auxiliary functions from $\mathcal{H}[\mathrm{M}]$ induce a formal C^*-algebra structure on $A[\mathrm{M}]$, with a notion of *adjointness* and a meaning of a *positive* eigenvalue.

B is generated by *self-adjoint* operators and M consists of positive eigenspaces (so elements of M may be called *states*).

The operators $U_\gamma \in A[\mathrm{M}]$ become *unitary* and act on B by conjugation. This corresponds to the action of the γ on M.

$A(\mathrm{M})$ for the ℓ-cover of the affine line ($\epsilon \in \mathbb{F}, \quad \epsilon^\ell = 1$):

$$\begin{aligned}
&\mathrm{HY} = \mathrm{YH}; \quad \mathrm{HZ} = \mathrm{ZH};\\
&\mathrm{YZ} = \mathrm{ZY}; \quad \mathrm{Y}^\ell = I; \quad \mathrm{Z}^\ell = I;\\
&\mathrm{UH} - \mathrm{HU} = h\mathrm{U}; \quad \mathrm{VH} - \mathrm{HV} = ih\mathrm{V};\\
&\mathrm{UY} = \epsilon\mathrm{YU}; \quad \mathrm{YV} = \mathrm{VY};\\
&\mathrm{ZU} = \mathrm{UZ}; \quad \mathrm{VZ} = \mathrm{YZV};\\
&\mathrm{E} = \mathrm{U}^{-1}\mathrm{V}^{-1}\mathrm{UV}; \quad \mathrm{E}^\ell = I;\\
&\mathrm{UE} = \mathrm{EU}; \quad \mathrm{VE} = \mathrm{EV}.
\end{aligned}$$

$\mathrm{Y}, \mathrm{Z}, \mathrm{U}, \mathrm{V}$ and E unitary, H self-adjoint (slightly simplified).

Inverse problem. Start with a noncommutative algebra A and produce a Zariski $\mathrm{M} = \mathrm{M}[A]$.

Quantum algebras at roots of unity

We assume for a "quantum algebra A at roots of unity":

1. A is an affine unital $\mathbb{F}$-algebra, finite-dimensional over its centre $Z(A)$. $\mathbb{F}$ algebraically closed.

2. Isomorphism classes of generic irreducible A-modules are in a bijective correspondence with an open subset $V^0 \subseteq \operatorname{Max} Z(A)$ of the affine variety.

3. Generic irreducible modules allow a uniform choice of canonical bases degenerating regularly outside V^0 preserving the dimension.

Examples (for $q^\ell = 1$)

1. $A = \langle U, V : UV = qVU \rangle$ Manin's quantum plane

2. $A = U_q(\mathfrak{sl}_2)$ quantised $U_q(\mathfrak{sl}_2)$ as a Hopf algebra (quantum group)

3. $A = O_q(\mathrm{SL}_2)$ quantised co-ordinate Hopf algebra of SL_2 (quantum group)

...

We associate with every such A the bundle

$$\mathrm{mod}_A^{(\ell)} = \{\mathfrak{m}_a : a \in \mathrm{Max}\, Z(A)\}$$

of ℓ-dimensional A-modules $\mathfrak{m}_a$ (with or without selected canonical bases) over the algebraic variety $V_A = \mathrm{Max}\, Z(A)$.

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Typically, other finite-dimensional A-modules as well as morphism maps between modules are definable in $\mathrm{mod}_A^{(\ell)}$, so we expect that up to interdefinability *the structure* $\mathrm{mod}_A^{(\ell)}$ *is equivalent to* mod_A, *the category of finite dimensional* A*-modules.*

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One may also consider the infinite-dimensional A-module

$$\mathcal{H} := \sum_a \mathfrak{m}_a$$

with or without a choice of canonical bases in each $\mathfrak{m}_a$.

Theorem *The structure* $\mathrm{mod}_A^{(\ell)}$ *is a Zariski geometry, with respect to a Zariski topology.*

The $\mathbb{F}$*-algebra* A *is determined by* $\mathrm{mod}_A^{(\ell)}$ *as the algebra of definable linear transformations of* $\mathcal{H}$ *(equivalently, of the vector bundle* $\mathfrak{m}_a$*).*

$\mathrm{mod}_A^{(\ell)}$ *is not definable in commutative algebraic geometry, in general.*

For A *commutative,* $\mathrm{mod}_A^{(\ell)}$ *is the trivial line bundle over* $\mathrm{Max}\,A$, *and so the geometry is equivalent to that of the algebraic variety* $\mathrm{Max}\,A$, .

Remark $\mathrm{mod}_A^{(\ell)}$ is not the unique construction satisfying the properties above. Other constructions produce *definably equivalent* Zariski geometries. In all the cases (known to us) these are equivalent to mod_A, the category of finite dimensional A-modules.

Not a root of unity case.

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The quantum harmonic oscillator

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$$A = \langle \mathrm{P},\ \mathrm{Q} :\ \mathrm{PQ} - \mathrm{QP} = ih \rangle$$

as C^*-algebra.
In mathematical physics

$$\mathrm{H} = \frac{1}{2}(\mathrm{P}^2 + \mathrm{Q}^2),$$

the Hamiltonian of the harmonic oscillator.

$\mathrm{P},\ \mathrm{Q}$ and H are self-adjoint.

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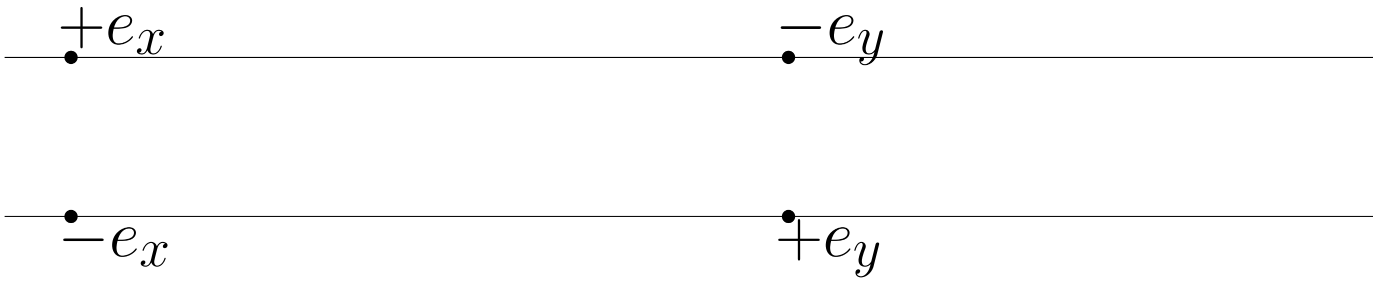
$$\mathrm{C}_+ = \frac{1}{2}(\mathrm{P} + i\mathrm{Q}), \quad \mathrm{C}_- = \frac{1}{2}(\mathrm{P} - i\mathrm{Q})$$

the creation and annihilation operators;

$$\mathrm{C}_+\mathrm{C}_- = \mathrm{H} + \frac{h}{2},\ \ \mathrm{C}_-\mathrm{C}_+ = \mathrm{H} - \frac{h}{2},\ \ \mathrm{C}_+\mathrm{C}_- - \mathrm{C}_-\mathrm{C}_+ =$$

The universe $E = \{\pm e_x : \; x \in \mathbb{C}\}$



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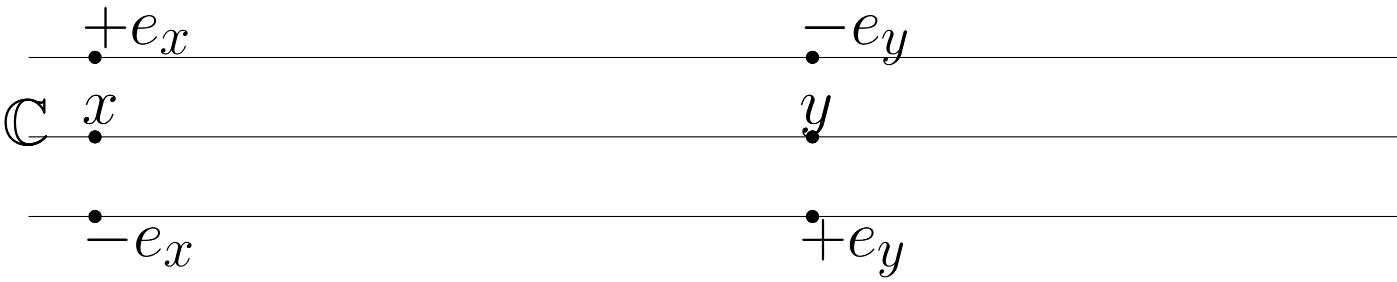

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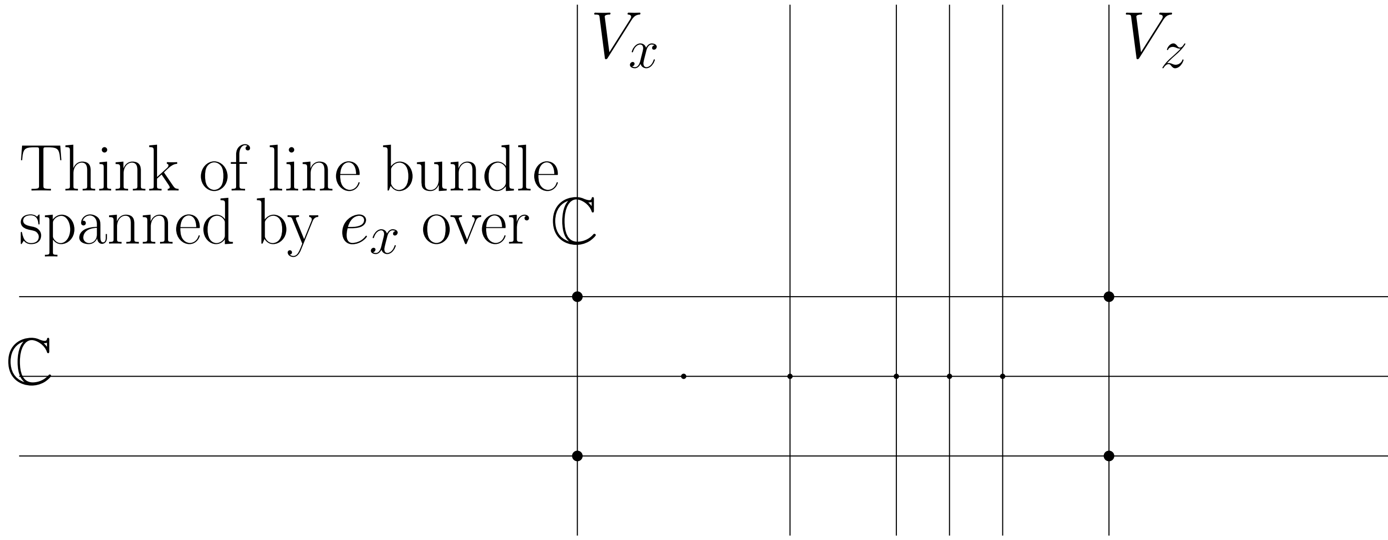
Think of line bundle spanned by e_x over $\mathbb{C}$



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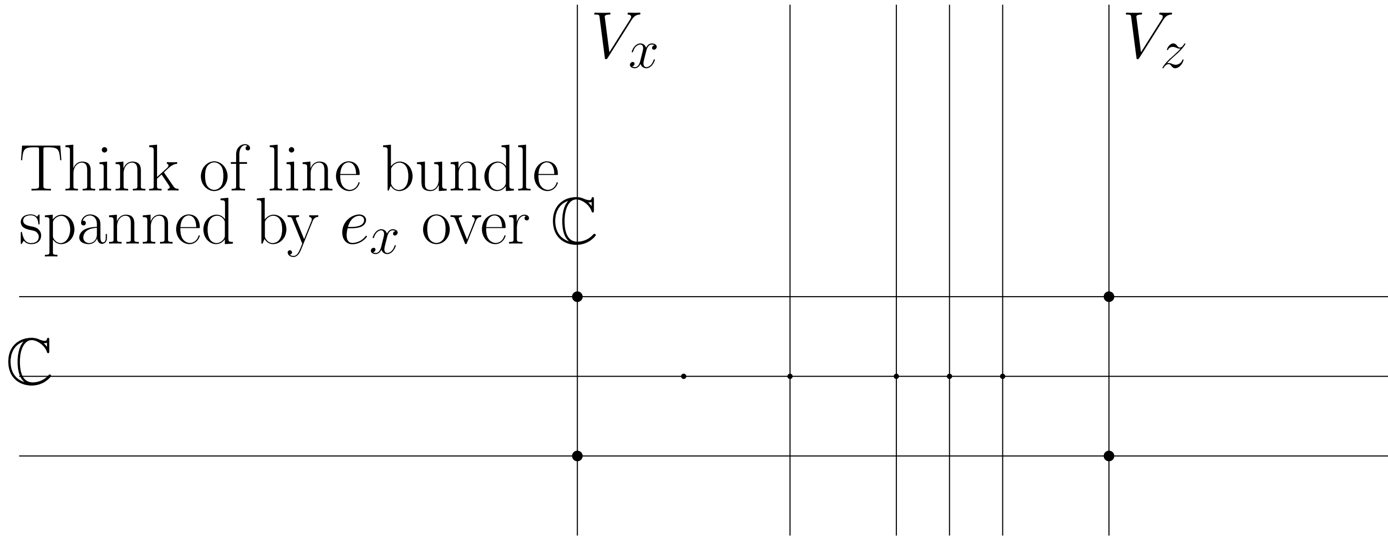

$\mathrm{H} + \frac{1}{2}$ defines the linear maps
$V_x \longrightarrow V_x; \quad v \mapsto x\,v$

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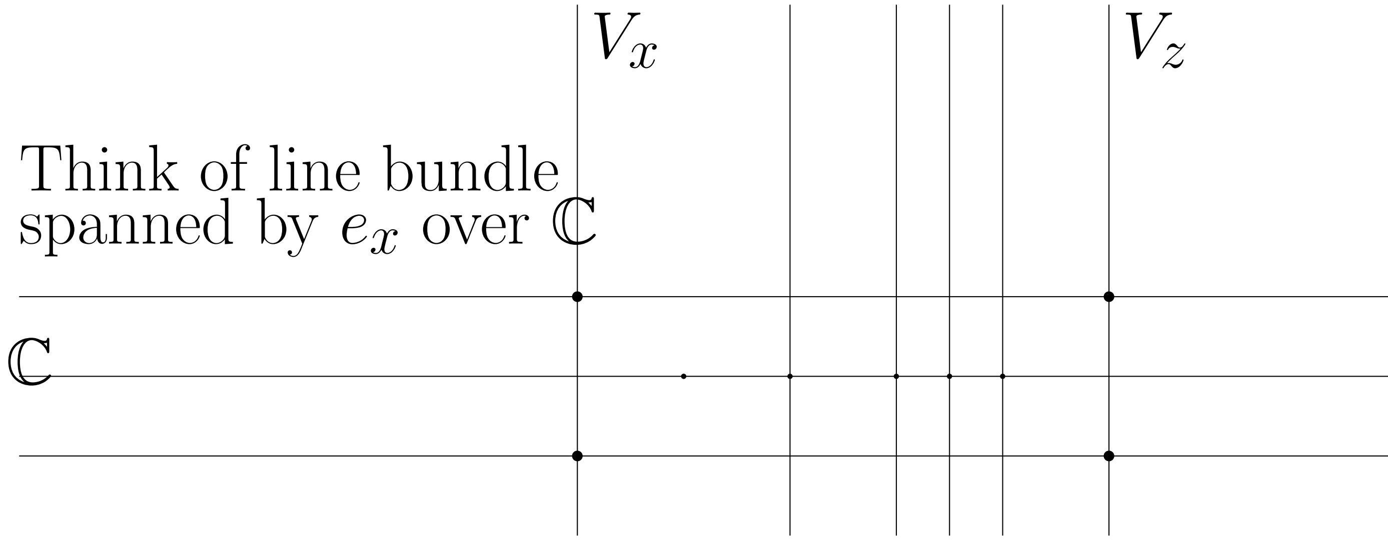
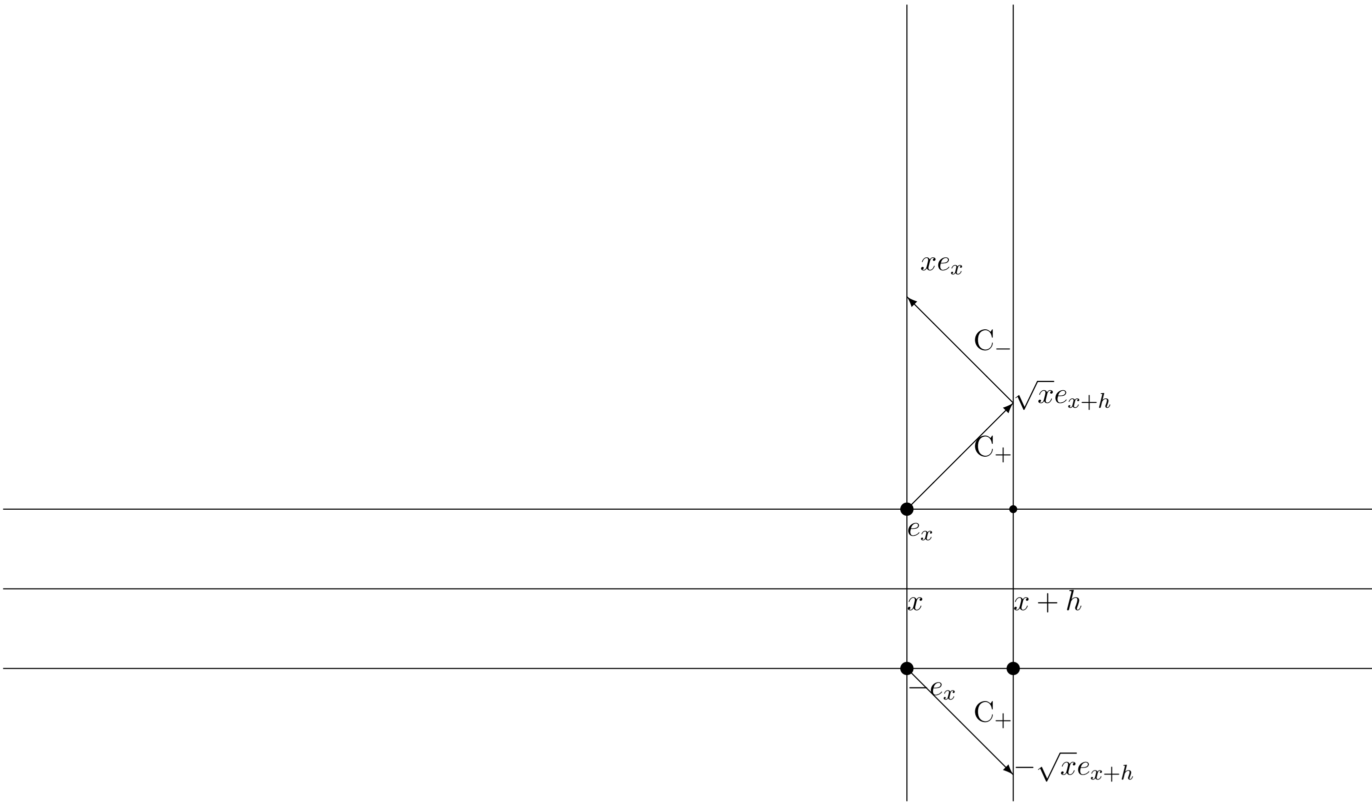


C$_+$ and C$_-$ define linear maps

C$_+ : V_x \longrightarrow V_{x+h}$

C$_- : V_x \longrightarrow V_{x-h}$



C$_+$ and C$_-$ define linear maps

$$\mathrm{C}_+ : V_x \longrightarrow V_{x+h}; \quad \lambda e_x \mapsto \sqrt{x}\lambda e_{x+h}$$

$$\mathrm{C}_- : V_x \longrightarrow V_{x-h}; \quad \lambda e_x \mapsto \sqrt{x-h}\lambda e_{x-h}$$

Theorem The structure $\mathrm{E}(A)$ corresponding to the quantum harmonic oscillator is a 1-dimensional (complex) Zariski geometry.

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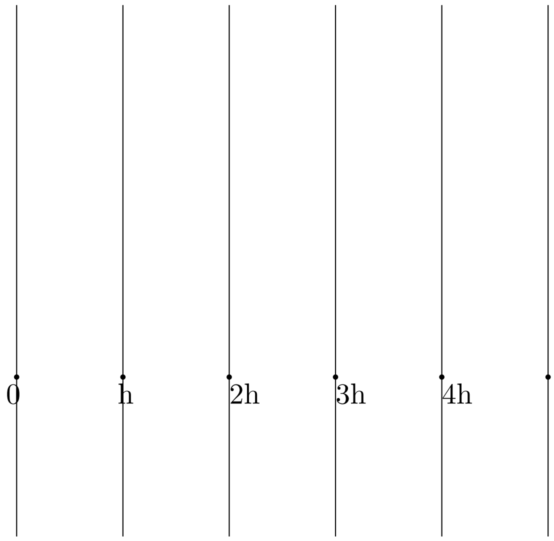
When one applies the full restrictions imposed by the C^*-algebra structure one gets **the real part** of $\mathrm{E}(A)$, which is discrete in this case.



Problem Explain model-theoretically transitions between bases of H-eigenvectors, P-eigenvectors and Q-eigenvectors.

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Over the field of characteristic p the algebra A is a "quantum algebra at roots of unity" and so $\mathrm{M}(A)$ is a Zariski geometry again.

Problems and projects

1. Establish a right category of geometric objects corresponding to non-commutative algebras A :

- as "algebro-geometric" coordinate algebras,

- as C^*-algebras,

- understand the interplay of the algebro-geometric and real geometric structures.

2. Develop a deformation (approximation) theory at the level of geometric objects

- e.g. as $h \to 0$

- as a root of unity converges to a generic q

- to explain how (and if) an elliptic curve deforms into a quantum torus.

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3. Explain model-theoretically the meaning of various non-convergent sums of maths physics.

Example.

Theorem There is a well-defined Gromov-Hausdorff limit of the ℓ-cover of the affine line

$$\lim_{\frac{1}{\ell}=h\to 0} \mathrm{M}_h = \begin{cases} \text{real differentiable manifold} = \\ U(1)\text{-gauge field over a 2-dim real m} \end{cases}$$

The limit of the unitary operators U and V correspond to covariant differentiation on the gauge field. For the ℓ-cover of the torus $\mathbb{C}^*$, the connection of the gauge field is of non-constant curvature.