# Geometric dualities and model theory

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# Tarskian duality

$$\text{Theory } T \longleftrightarrow \text{Class of models } \mathfrak{M}(T)$$

For a $\kappa$-categorical $T$

$$\text{Theory } T \longleftrightarrow \text{Model } M_T \text{ (of cardinality } \kappa)$$

# Geometric dualities

Affine commutative $\mathbb{C}$-algebra

$$R = \mathbb{C}[X_1, \ldots, X_n]/I$$

Complex algebraic variety

$$\mathbf{V}_R$$

Commutative unital $C^*$-algebra

$$A$$

Compact topological space

$$\mathbf{V}_A$$

Affine $k$-scheme

$$R = k[X_1, \ldots, X_n]/I$$

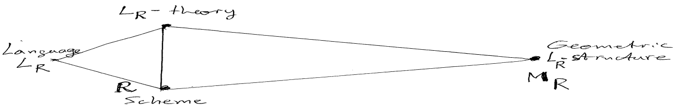
The geometry of $k$-definable points, curves etc of an algebraic variety $\mathbf{V}_R$

$k$-scheme of finite type

$$S$$

The geometry of $k$-definable points, curves etc of a “Zariski geometry” $\mathbf{V}_S$

# From the scheme – language – theory to the structure



# Claim A

These are syntax – semantics dualities.

The dualities can be recast in the form of Tarskian dualities.
In general the syntax may come with a topology (as in $C^*$-algebras).
Recall also the syntax of *continuous model theory*.

# What is geometric semantics?

Non-example. The models of the theory of arithmetic **do not** provide a semantics of geometric type.

**Suggestions.** The structures on the right hand side of dualities must be stable (in a generalised sense).

There also should be a topology associated in a natural way with the syntax.

This leads again to the notion of Zariski geometry.

# $L$-Zariski geometries.

**V** is said to be a Noetherian $L$-Zariski if it satisfies

- Closed subsets of $V^n$ are exactly those which are $L$-positive-quantifier-free definable.
  *(There may be points which are not closed!)*
- The projection of a closed set is constructible (positive quantifier-elimination).
- A good dimension notion on closed subsets is given. The addition formula and the fibre conditions hold.
- For every extension $L(C)$ of the language by constants, any $\mathbf{V}' \succeq \mathbf{V}$ as an $L(C)$-structure satisfies the above.

**Theorem.** Noetherian $L$-Zariski geometries are of finite Morley rank.

# Example. Affine $k$-schemes.

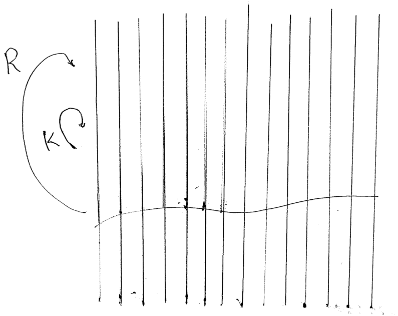
Given a field $k$ and an affine $k$-algebra $R$ we introduce a language $L_R$ and an $L_R$-Zariski structure $\mathbf{V}_R$.

**The language** $L_R$ of two-sorted structures:

- sort $K$ for an algebraically closed field $K$ containing $k$ with names for elements of $k$;
- sort $V$ fibred as the union of all irreducible (1-dimensional) representations of $R$;
- $L_R$ has means to describe the action of the field $K$ and the additive structure on fibres of $V$;
- $L_R$ has a name $a$ for each $a \in R$ interpreted as a linear operator on each fibre (each vector space) of $V$.

The structure $\mathbf{V}_R$ has exactly one fibre for each isomorphism type of irreducible representations.

# Illustration: sort $V$ and a section



We define natural morphisms between the $L_R$-structures (for different $R$).

**Duality Theorem.** *The category of affine $k$-algebras for all $k \subseteq K = K^{alg}$ is isomorphic to the category of respective Zariski stuctures.*
Points (non necessarily closed) of $\mathbf{V}_R$ correspond to irreducible $K$-representations of $R$.

# A noncommutative duality Theorem

The above duality can be extended to non-commutative geometry “at roots of unity”.

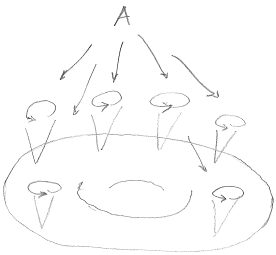
$$A_{\mathbf{V}} \longleftrightarrow \mathbf{V}_A.$$

$A_{\mathbf{V}}$ – co-ordinate algebra, $\mathbf{V}_A$ – Zariski geometry.

# A non-commutative example “at root of unity”

Non-commutative 2-torus at $q = e^{2\pi i \frac{m}{N}}$ has co-ordinate ring $A = A_q =$
$\langle U, V : \; U^* = U^{-1}, \; V^* = V^{-1}, \; UV = qVU \rangle$

Points have structure of (an orthonormal basis) of a $N$-dim Hilbert space.



| | |
|---|---|
| Affine commutative $\mathbb{C}$-algebra $R$ | Complex algebraic variety $\mathbf{V}_R$ |
| Commutative $C^*$-algebra $A$ | Compact topological space $\mathbf{V}_A$ |
| Affine $k$-scheme $R$ | The $k$-definable structure on an algebraic variety $\mathbf{V}_R$ |
| $k$-scheme of finite type $S$ | The $k$-definable structure on a Zariski geometry $\mathbf{V}_S$ |
| $C^*$-algebra $A$ at roots of unity | Zariski geometry $\mathbf{V}_A$ |
| Weyl-Heisenberg algebra $\langle Q, P : \ QP - PQ = i\hbar \rangle$ | ? |
| The integers $\mathbb{Z}$ | ? |

# $QP - PQ = i\hbar$

“The whole of quantum mechanics is in this **canonical commutation relation**”.

An analogy:

$$Y = X^2 + aX + b$$

is (the equation of) a parabola.

$$H = \frac{1}{2}(P^2 + \omega^2 Q^2)$$

is (the Hamiltonian of) a quantum harmonic oscillator.

# $QP - PQ = i\hbar$

This does not allow the $C^*$-algebra (Banach algebra) setting. On suggestion of Herman Weyl and following Stone – von Neumann Theorem replace the Weyl-Heisenberg algebra by the category of **Weyl $*$-algebras**

$$A_{a,b} = \left\langle U^a, V^b : \; U^a V^b = e^{iab\hbar} V^b U^a \right\rangle,$$

$$a, b \in \mathbb{R}, \; U^a = e^{iaQ}, \; V^b = e^{ibP}.$$

where it is also assumed that $U^a$ and $V^b$ are unitary.

We may assume that $\frac{\hbar}{2\pi} \in \mathbb{Q}$ and so, when $a, b \in \mathbb{Q}$ the algebra $A_{a,b}$ is at root of unity. We call such algebras **rational Weyl algebras.**

# Sheaf of Zariski geometries over the category of rational Weyl algebras

The category $\mathcal{A}_{\rm fin}$ has objects $A_{a,b}$, rational Weyl algebras, and morphisms = embeddings.
Note,

$$A_{a,b} \subseteq A_{c,d} \text{ iff } cn = a \;\&\; dm = b \text{ for } n, m \in \mathbb{Z} \setminus \{0\}.$$

This corresponds to the surjective morphism in the dual category $\mathcal{V}_{\rm fin}$ of Zariski geometries

$$\mathbf{V}_{A_{a,b}} \to \mathbf{V}_{A_{c,d}}.$$

The duality functor

$$A \;\mapsto\; \mathbf{V}_A$$

can be interpreted as defining a **sheaf of Zariski geometries** over the category of rational Weyl algebras.

# Completions of $\mathcal{A}_{\rm fin}$ and $\mathcal{V}_{\rm fin}$.

The completion of $\mathcal{A}_{\rm fin}$ is $\mathcal{A}$, the category of all Weyl algebras in the Lie-groups topology (not the Banach algebra topology):

$$\sigma U^\alpha V^\beta \to s U^a V^b \text{ iff } \sigma \to s,\ \ \alpha \to a \text{ and } \beta \to b,$$

$$\sigma, \alpha, \beta \in \mathbb{Q},\ \ s, a, b \in \mathbb{R}$$

Completing $\mathcal{V}_{\rm fin}$ is the main difficulty of the project.

We use **structural approximation**, which in basic cases is equivalent to the ultraproduct construction of continuous model theory.

**Exercise.** Use the ultraproduct of continuous model theory to construct the universal cover of the algebraic torus $\mathbb{C}^\times$.

# The space of states.

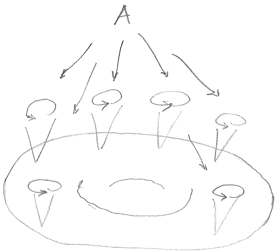
We construct a $\mathcal{V}_{\rm fin}$**-projective** limit object $\mathbf{V}_{\mathcal{A}}$, corresponding to the algebra $\bigcup \mathcal{A}$.
By construction

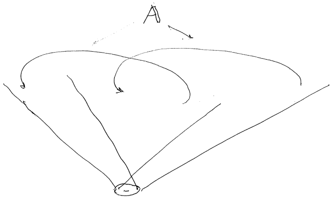
$$A \in \mathcal{A}_{\rm fin} \quad \Rightarrow \quad \mathbf{V}_{\mathcal{A}} \twoheadrightarrow \mathbf{V}_{A}.$$

This object is closely related to *the Hilbert space of quantum mechanics*. We call the object **the space of states.**

**Remark.** The same construction for the category of commutative algebras $\langle U^a, U^{-a}\rangle, \ \ a \in \mathbb{Q}$, (equivalent to the category of etalé covers of $\mathbb{C}^\times$) produces the universal cover $(\mathbb{C}, +)$ of $\mathbb{C}^\times$ and the morphism $\exp : \mathbb{C} \to \mathbb{C}^\times$.

## How noncommutative $\mathbf{V}_A$ deforms into $\mathbf{V}_{\mathcal{A}}$.



# Operators $\mathrm{P}, \mathrm{Q}$ on $\mathbf{V}_{\mathcal{A}}$.

We define in each member of the ultraproduct

$$\mathrm{Q} := \frac{U^a - U^{-a}}{2ia}, \quad \mathrm{P} := \frac{V^b - V^{-b}}{2ib}$$

in accordance with

$$U^a = e^{ia\mathrm{Q}}, \quad V^b = e^{ib\mathrm{P}}.$$

Then in the limit, we can calculate in $\mathbf{V}_{\mathcal{A}}$, for any state $e$

$$(\mathrm{QP} - \mathrm{PQ})e = i\hbar e.$$

So, **in the space of states:**

$$\mathrm{QP} - \mathrm{PQ} = i\hbar I.$$

In other words, there is a Lie algebra $\langle \mathrm{Q}, \mathrm{P} \rangle$ acting on $\mathbf{V}_{\mathcal{A}}$ (the Heisenberg algebra).

# Time evolution operators on the space of states

**Theorem.** *Automorphisms of $\mathcal{A}_{fin}$ give rise to certain operators on $\mathbf{V}_{\mathcal{A}}$. These are definable in the sense of continuous model theory.*

Such operators $K$ $(= K_{\text{particle}})$ are typically the "time evolution operators for a given particle".

The one-parameter subgroups $\{K^t : t \in \mathbb{R}\}$ describe the time evolution of *the particle corresponding to* $K$.

# Restriction to a commutative algebra case

**Theorem.** A “time evolution operator” on $(\mathbb{C},+)$ (as the cover of $\mathbb{C}^\times$) has the form

$$z \mapsto \kappa z, \quad \text{some } \kappa \in \mathbb{R}_{>0}.$$

This corresponds to ’raising to power’ $\kappa$ on $\mathbb{C}^\times$.

Recall: the theory of raising to real power is superstable, provided Schanuel’s conjecture is true.

# Example. Quantum harmonic oscillator.

The Hamiltonian:

$$H = \frac{1}{2}(P^2 + Q^2)$$

The time evolution operator :

$$K^t = K^t_H := e^{-i\frac{H}{\hbar}t}, \quad t \in \mathbb{R}.$$

This “induces” the automorphism of the category of algebras

$$U^a \mapsto e^{-\frac{2\pi a^2 \sin t \cos t}{2}} U^{a \sin t} V^{a \cos t}$$
$$V^a \mapsto e^{\frac{2\pi a^2 \sin t \cos t}{2}} U^{-a \cos t} V^{a \sin t}$$

# Scheme of calculations

- rewrite the formula over $\mathbf{V}_{\mathcal{A}}$ in terms of Zariski-regular pseudo-finite sums and products over $\mathbf{V}_A$, $A \in \mathcal{A}_{fin}$;
- calculate **uniformly** in $\mathbf{V}_A$ (using e.g. the *Gauss quadratic sums* formula)
- apply ultraproduct (continuous model theory) to the result and get the result in terms of the standard reals.

# Example. Quantum harmonic oscillator.

$$K^t = K_H^t := e^{-i\frac{H}{\hbar}t}, \quad t \in \mathbb{R}.$$

To calculate $K^t$ we *approximate* assuming $\sin t, \cos t \in \mathbb{Q}$. This transfers us to the rational category $\mathcal{V}_{fin}$ and to the calculations in (pseudo)finite-dimensional spaces.

Then the matrix element on row $x_1$ and column $x_2$ ( *kernel of the Feynman propagator*) is calculated as

$$\langle x_1 | K^t x_2 \rangle = \sqrt{\frac{1}{2\pi i\hbar \sin t}} \exp i \frac{(x_1^2 + x_2^2)\cos t - 2x_1x_2}{2\hbar \sin t}.$$

The trace of $K^t$,

$$\mathrm{Tr}(K^t) = \int_{\mathbb{R}} \langle x | K^t x \rangle = \frac{1}{\sin \frac{t}{2}}.$$

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Note that in terms of conventional mathematical physics we have calculated

$$\mathrm{Tr}(K^t) = \sum_{n=0}^{\infty} e^{-it(n+\frac{1}{2})},$$

a non-convergent infinite sum.

# Work in progress

Given a Hamiltonian in the form

$$H_f = \frac{\mathrm{P}^2}{2\hbar} + f(\mathrm{Q}), \quad f(X) \in \mathbb{Q}[X]$$

one can calculate using the approximation method and setting $\Delta t = \frac{1}{\nu}$, $\nu$ a big non-standard integer,

$$K_f^{\Delta t} \approx \mathrm{e}^{-i\frac{\mathrm{P}^2}{2\hbar}\Delta t} \cdot \mathrm{e}^{-if(\mathrm{Q})\Delta t}.$$

Both operators on the right **are definable.**
Given $t \in \mathbb{R}, \ t \approx \frac{\kappa}{\nu}$ it is natural to identify the time evolution operator for $H_f$ on $\mathbf{V}_{\mathbb{A}}$ as

$$K_f^t := \left(\mathrm{e}^{-i\frac{\mathrm{P}^2}{2\hbar\nu}}\mathrm{e}^{-i\frac{f(\mathrm{Q})}{\nu}}\right)^{\kappa}$$

(cf the Trotter product formula).

$$K_f^t := \left( e^{-i\frac{P^2}{2\hbar\nu}} e^{-i\frac{f(Q)}{\nu}} \right)^{\kappa}$$

Is it a regular transformation? I.e. is the image of the operator $K_f^t$ (under standard part map) well-defined in the space of states?

Equivalently, does

$$\lim \left( e^{-i\frac{P^2}{2\hbar\nu}} e^{-i\frac{f(Q)}{\nu}} \right)^{\kappa}$$

exist?

This is equivalent to the Feynman path integral hypothesis for QM.