Approximation, domination and integration

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B.Zilber, *The semantics of the canonical commutation relations* arxiv.org/abs/1604.07745

———-, *Structural approximation and quadratic quantum mechanics* In preparation

Problems with limits

How to speak about limit(s) of structures?

In particular, what limits do physicists have in mind?

The passage from discrete (**finite**) to continuous.

Sub-problem. “Very large” finite structure

The relevant formalism - **pseudo-finite** structures.

Determining a limit

All structures are in a language L.

Topology. We consider a topology on M^n, all n the **base of closed subsets** of which are the **quantifier-free positively definable subsets** of the structures.

Definition. An L-structure **M** is a limit of a (pseudo-finite) L-structure N if there is a surjective L-homomorphism

$$\lim : \mathrm{N} \twoheadrightarrow \mathbf{M}.$$

In the case

$$\mathrm{N} = \prod_{i \in I} \mathrm{N}_i / \mathcal{D}$$

we can also write equivalently

$$\lim_{\mathcal{D}} : \mathrm{N}_i \twoheadrightarrow \mathbf{M}$$

Determining a limit

The definition of limit is well-behaved, in particular,

$$\lim_{\mathcal{D}} : \mathrm{N}_i \to \mathbf{M}, \text{ for all } \mathrm{N}_i = \mathbf{M},$$

when **M** is *compact*, i.e.

- the intersection of any filter of closed subsets of $\mathbf{M}^n$ is non-empty;
- the image of a closed $S \subset \mathbf{M}^{n+1}$ under projection $\mathbf{M}^{n+1} \to \mathbf{M}^n$ is closed.

I.e. $\exists$-positive definable sets are closed, that is are positive q-f definable.

Approximation of fields and rings

Any locally compact field (such as $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Q}_p$) can be *compactified* by adding ∞.
$\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, $\bar{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$, ...

Theorem.
1. The compactified field $\bar{\mathbb{C}}$ (or any other ACF) is approximable by finite fields.

2. The rings $\mathbb{Z}_p$ are approximable by finite rings $\mathbb{Z}/p^n$.

3. The compact subring $\hat{\mathbb{Z}} = \prod \mathbb{Z}_p$ of adeles is approximable by finite rings $\mathbb{Z}/N$, for N "highly divisible".

4. The compactified field $\bar{\mathbb{R}}$ is NOT approximable by finite fields or finite rings. Neither are any other locally compact fields.

Some positive model theory notions

We say, in the context of lim : $\mathrm{N} \twoheadrightarrow \mathbf{M}$,

$$\mathrm{N} \equiv_{Pos} \mathbf{M}$$

if for T_i, the **inductive theory**,

$$\mathrm{N} \vDash T_i(\mathbf{M}).$$

(compare with Ben-Yaacov and Poizat, 2007).
We say

$$\mathrm{N} \succ_{Pos} \mathbf{M}$$

if there are $L(M)$- expansions

$$\mathrm{N}^\sharp \equiv_{Pos} \mathbf{M}^\sharp$$

In this case there exists

$$\mathrm{colim} : \mathbf{M} \hookrightarrow \mathrm{N}$$

An approximation lim which possesses colim we call **strong**.

Positive interpretability works well under strong approximation.

Examples

1. The compactification of any alg.closed field is approximable by finite fields. It is **strong** if the pseudo-finite field contains zeroes of all polynomials over $\mathbb{Z}$ and is enough saturated.

2. The rings $\mathbb{Z}_p$ are approximable by finite rings $\mathbb{Z}/p^n$. **Strong.**

3. The compact subring $\hat{\mathbb{Z}} = \prod \mathbb{Z}_p$ of adeles is approximable by finite rings $\mathbb{Z}/N$, for N "highly divisible". **Strong.**

4. How to approximate $\mathbb{R}$?

Weak ring structure on $\bar{\mathbb{R}}$

The "weak ring structure" $\mathbb{R}_{\mathrm{w}}$ on $\mathbb{R}$ is given by $+, \leq$, the family of unary operations $x \mapsto r \cdot x$, all $r \in \mathbb{R}$, and the 4-ary relation

$$\mathcal{P}(x_1, y_1, x_2, y_2) :\equiv (x_1 y_1 - x_2 y_2) \in \mathrm{h} \cdot \mathbb{Z}$$

for an $\mathrm{h} \in \mathbb{R}_{>0}$.
There is an obvious extension (compactification) of the weak structure to $\bar{\mathbb{R}}$.

Note. The weak structure on $\bar{\mathbb{R}}$ allows to recover the usual ring structure on $\mathbb{R}$, if one uses all the power of first-order logic. But, *the ring structure is not recoverable by positive formulas*.

Weak ring structure on $\mathbb{Z}/N$

This is given as

$$(\mathbb{Z}/N; +, \leq, \mathcal{P}, nx = my)_{\frac{m}{n}=r}$$

where, $\mathcal{P}(x_1, y_1, x_2, y_2) :\equiv (x_1 y_1 = x_2 y_2)$ and $\leq$ is the cyclic order.

Theorem. *Given a non-standard integer η of the form*

$$\eta = \nu^2 \mathrm{h} : \quad \nu \text{ “highly divisible”}, \ \ \mathrm{h} = \frac{m}{n}, \ \ m, n << \nu$$

there is a “well-behaved”

$$\lim : \ {}^*\mathbb{Z}/\eta \twoheadrightarrow \bar{\mathbb{R}}_w.$$

Question. Is the lim strong for unordered $\bar{\mathbb{R}}_{\mathrm{w}}$?

Structures over $\mathbb{R}_{\mathrm{w}}$

Theorem. The finite approximation of $\bar{\mathbb{R}}_{\mathrm{w}}$ can be extended to a finite approximation of the mathematical structure of quantum mechanics with quadratic Hamiltonians:

the symplectic space on $\mathbb{R}^{2n}$ with Hermitian line bundle with connection. The metaplectic group $\mathrm{Mp}(n)$ of symplectomorphisms (= time evolution operators) acts (element-by-element) on the space and on the line bundle. Its 1-dim subgroups describe the evolution of respective "particles".

Calculations over finite models can be passed via **lim** to the continuous model of quantum mechanics.

Comparison with Fourier analysis over finite groups

There are numerous observations and remarks concerning the **similarity** of a Fourier analysis over $\mathbb{Z}/N$ and the calculus of quantum mechanics.
Also relevant is the “discrete Dirac calculus”.

See also S.Albeverio, Y,Gordon and A.Khrennikov, 2000

Our statement is about the **full model M** over $\mathbb{R}_w$ with arbitrary realistic parameters ($\hbar, \omega, t$ ...) and approximation understood in a much broader sense.

Measure

We say that a collection Δ of constructible sets in **M** is **measurable via approximation** lim : $\mathrm{N} \to$ **M** if there is non-standard integer ν such that for every constructible $\Psi \in \Delta$ in n variables and every $\epsilon > 0$ there are a positive real number m_Ψ and constructible Ψ^-_ϵ and Ψ^+_ϵ such that,

$$\Psi^-_\epsilon(\mathrm{N}) \subseteq \lim^{-1}\Psi(\mathbf{M}) \subseteq \Psi^-_\epsilon(\mathrm{N})$$

and

$$m\Psi - \epsilon \leq \nu^{-n}|\Psi^-_\epsilon(\mathrm{N})| \quad \text{and} \quad \nu^{-n}|\Psi^+_\epsilon(\mathrm{N})| \leq m\Psi + \epsilon.$$

Thus for Δ-sets in **M** is assigned dimension n and a **measure** $m\Psi$, which is obviously additive.

Bounded sets

We allow **M** to have finitely many *singular points* $\infty_1, \ldots, \infty_n$ such that for any closed $\Phi(\mathbf{M})$

$$\infty_i \in \Phi(\mathbf{M}) \subset \mathbf{M} \quad \Rightarrow \quad \nu^{-1}|\Phi(\mathrm{N})| > r, \text{ for every } r > 0.$$

We say that $\Psi \subset \mathbf{M}^n$ is **bounded** if no projection on **M** of the closure $\mathrm{pr}\bar{\Psi}(\mathbf{M}) \subset \mathbf{M}$ contains ∞_i.

We say that **M** is measurable via approximation $\lim : \mathrm{N} \to \mathbf{M}$ if the family of constructible bounded sets in **M** is measurable. The measure is **rich** if for every bounded open $\Psi(\mathbf{M}) \subset \mathbf{M}^n$,

$$0 < m_\Psi < \infty.$$

Also, note that our definition treats **M** as one-dimensional structure.

Measure of boundaries and compact domination

The necessary condition for existence of measure in **M** is the condition of **compact domination**: *the measure of boundaries of bounded sets is equal to 0.*

(cf. Pillay, Hrushovski, Peterzil, Berarducci, Simon ...)

Integration

Given a measurable structure with a qf-definable function $f : \mathbf{M}^n \to \mathbb{C}$ (so $f : \mathrm{N}^n \to \mathbb{C}$ is also definable) we define an **integral**

$$\int_{\Psi(\mathbf{M})} f \, \mathrm{d}m = \lim \sum_{k \in \Psi(\mathrm{N})} f(k) \Delta k$$

where Δk is the measure of the point $k \in \Psi(\mathrm{N})$.

Warning. This is not equivalent to the Riemann or Lebesgue integration. Can be well-defined when the others are not.

Often, the difficulty is to determine Δk, which depends on Ψ.

The definition works for:

- p-adic and adelic integration;
- calculus of quantum mechanics (over $\bar{R}_w$).

Physicists calculations of path integral

$$I_N = \int_{\mathbb{R}} \dots \int_{\mathbb{R}} e^{i \sum a_{kj} x_k x_j} dx_1 \dots dx_N, \quad K^t = \lim_{N \to \infty} I_N$$

where N defines $\Delta t = \frac{t}{N}$ for which the evolution from t_k to $t_k + \Delta t$ is evaluated. I_N calculates the evolution from t_0 to $t_0 + t$. For the Harmonic oscillator,

$$|I_N| = \sqrt{N} \prod_{n=1}^{N} \sqrt{\left(\frac{\pi^2 n^2}{t^2} - \omega^2\right)^{-1}}$$

which requires a definition of “generalised limit” in order to get a right answer.

Structural approximation calculations of path integral

$$I_\kappa = \sum_{x_1=-\eta/2}^{\eta/2} \dots \sum_{x_\kappa=-\eta/2}^{\eta/2} e^{i\sum a_{kj}x_k x_j} \Delta x_1 \dots \Delta x_\kappa, \quad K^t = \lim I_\kappa$$

$0 << \kappa << \eta$. $\Delta x = c\frac{1}{\sqrt{\eta}}$, $\Delta t = \frac{t}{\kappa}$.
The control over the pseudo-finite parameters allows to evaluate K^t correctly.

In fact, the calculation is based on determining properly the **measure** $\Delta x_1 \dots \Delta x_\kappa$ **in pseudo-finite dimensional space of paths of length** κ.

Example of Calculation. Time evolution operator for the quantum harmonic oscillator.

$$K^t := e^{-i\frac{\mathrm{P}^2+\omega^2\mathrm{Q}^2}{\hbar}t}, \quad t \in \mathbb{R}.$$

where P and Q are the momentum and position operators, t time.

The matrix element on row x_1 and column x_2 (*kernel of the Feynman propagator*) is calculated as

$$\langle x_1 | K^t x_2 \rangle = \sqrt{\frac{\omega}{2\pi i\hbar \sin t}} \exp i\omega \frac{(x_1^2 + x_2^2)\cos t - 2x_1 x_2}{2\hbar \sin \omega t}.$$

Note that in terms of conventional mathematical physics we have calculated

$$\mathrm{Tr}(K^t) = \sum_{n=0}^{\infty} e^{-it(n+\frac{1}{2})},$$

a non-convergent infinite sum.