Beautiful pairs of valued fields and analytifications

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Berkovich analytification

Let K be an algebraically closed field and complete with respect to a non-Archimedean norm $|\cdot| : K \to \mathbb{R}_{\geq 0}$. One wishes to develop a theory of analytic functions on K as in $\mathbb{C}$.

Unlike $\mathbb{C}$, the non-Archimedean axiom implies that the topology on K is totally disconnected; Hence, the locally analytic functions behave quite arbitrarily.

Historically, Tate approached this by working with a Grothendieck topology. The resulting analytic spaces have a nice function theory, but lack topological intuitions.

Berkovich analytification

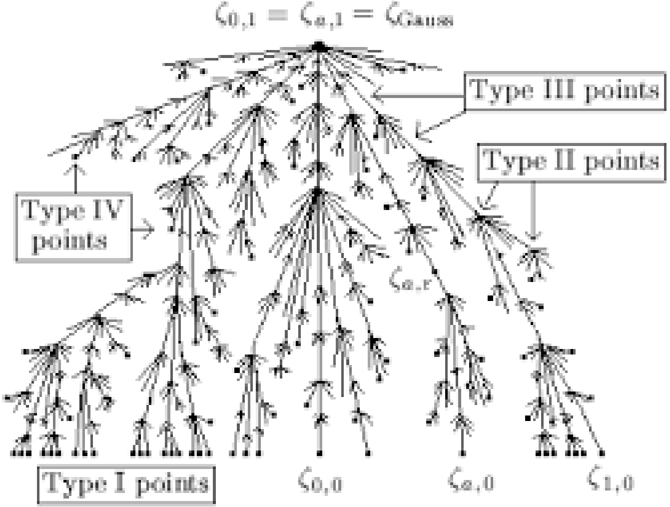
Berkovich approached the question by considering working with valuations.

Definition

Let K be as before and V an affine variety over K. Let V^{an} denote the set of multiplicative seminorms $K[V] \to \mathbb{R}_{\geq 0}$ extending that of K, and equipped with the weakest topology such for each $f \in K[V]$, the map $p \mapsto p(f)$ is continuous. V^{an} arises from gluing for general varieties.

V^{an} has nice topological properties: Hausdorff (if V is separated), locally compact, locally contractible.

- **Type I** Points $c \in K$.
- **Type II** Closed balls with center $c \in K$ and radius $r \in |K|$.
- **Type III** Closed balls with center $c \in K$ and radius $r \notin |K|$.
- **Type IV** A nested family of balls with trivial intersection (in K).



One can easily see the Berkovich unit disk is contractible via the map that keeps on enlarging the radius of a ball. On Type I points, $(c, t) \mapsto$ ball with radius t and center c.
How do we make sense of such spaces model theoretically, and in such a meaningful way so that the above map becomes a "definable" morphism in this category?

The functor $V \mapsto V^{an}$ is a subfunctor of $V \mapsto S_V(K)$ for $K \models \mathrm{ACVF}$. But one rarely talks about definability on type spaces. So some effort to develop a geometric theory on it has to be made. Working in ACVF (and from now on, I will switch to valuations from norms), Hrushovski and Loeser identified a special subset (set of generically stable types) $\widehat{V}$ of the type space as model theoretic analogues of V^{an}.
The first step of their work is to show that the set $\widehat{V}$ is pro-definable (a small projective limit of definable sets in ACVF) or even strict pro-definable (transition maps are surjective). This grants us back a wide range of tools from model theory.

Model theoretic adic space

There are other attempts on the theory of analytic spaces over non-Archimedean fields. A notable one was by Huber.
The goal of the talk is to introduce the pairs of valued fields as a framework to discuss analytifications in various context. We will establish a model theoretic analogue of adic spaces denoted by $\widetilde{V}$, the set of bounded/continuous types on V, in this setting. And talk about various liftings of the results by Hrushovski and Loeser.

Definable types and pro-definability

We fix a monster model $\mathbb{U}$ of a theory T and a definable set X. By *definable types* on X, we mean global types concentrating on X that are definable over some small model, i.e. there is a small model M such that for each formula $\varphi(x;y)$, the set $\{c \in \mathbb{U} : \varphi(x;c) \in p\}$ is M-definable.
And we say a type $p \in S_x(\mathbb{U})$ is *generically stable* if there is $M \preceq \mathbb{U}$ such that p is definable and finitely satisfiable over M.
Let p be a definable type, we use $c_{\varphi,p}$ to denote the canonical parameter of its φ-definition. The type p is determined by the sequence $(c_{\varphi,p})_\varphi$.
In general, if the φ-definition is uniform over p, meaning for each $\varphi(x,y)$ there is $\psi(y,z)$ such that for each p, the φ-definition of p is given by $\psi(y, c_{\varphi,p})$. Then we have established pro-definability.

Strict pro-definability

In general, pro-definable sets are still not well-behaved enough to study topology/geometry. We say a pro-definable set is *strict pro-definable* if the transition maps in the inverse limit are all surjective.

When we identify a subset C of definable types as a pro-definable set via the canonical parameters, strict pro-definabilty is equivalent to $\{c_{\varphi,p} : p \in C\}$ is definable for each φ.

In the case of $\widehat{V}$, it is well known that NIP theories have uniform definition for generically stable types, and Hrushovski and Loeser used stable domination to show the strictness.

How to generalize this?

Pairs of models and analytification

Another natural framework to study geometric spaces/ analytifications is through pairs of models.
Let K be a non-archimedean field as before, and let L be a spherically complete valued field extending K with value group $\mathbb{R}$. Moreover, we insist that the residue field extension is also proper.
For varieties V over K, (equivalence classes of) point in $V(L)$ can be identified as points in V^{an}. In the particular case of the affine line, the map $(c,t) \to$ "Generic point of $\bar{B}(c,t)$" is "definable" in the language of pairs.
But this is not the right context to talk about $\widehat{V}$ or $\widetilde{V}$. Particularly, it is hard to understand the definable sets in such pairs. The natural thing to do is to understand "rich" pairs associated to the functors $V \mapsto \widehat{V}$ or $\widetilde{V}$.

Let us look at some specific examples.
Poizat first initiated the study of (proper) pairs of stable structures, and the rich ones are the "beautiful pairs". They are pairs of models $M \preceq N \models T$ such that M is sufficiently saturated and N is $|M|^+$-saturated.
All such pairs are elementarily equivalent, and assuming further technical property (nfcp), one gets that (M, N) is ω-saturated in $\mathcal{L}_P$ and M is a embedded as a pure substructure. In this case, for each $\mathcal{L}$-formula $\varphi(x, y)$, let $\psi(y, z)$ be the uniform definition, the set

$$\{c \in M : \exists b \in N \, \forall y \in M \, \varphi(b, y) \Leftrightarrow \psi(y, c)\}$$

is $\mathcal{L}$-definable in M, which is exactly the strictness we wanted. In the concrete case of ACF, we have that varieties (as a scheme) are strict pro-definable.

Pairs of models and analytifications

Along a similar direction, van den Dries and Lewenberg established that for any proper pairs of $K \preceq L \models \mathrm{RCF}$ such that K is Dedekind complete in L, they are all elementarily equivalent. Moreover, K is embedded as a pure substructure. This gives us the strictness of the set of definable types in real closed fields. In other words, the real spectrum of a variety over $\mathbb{R}$ is strict pro-definable. Now we introduce a general framewrok.

Pairial structures and $\mathcal{C}$-beautiful pairs

Fix a complete $\mathcal{L}$-theory T and $\mathcal{C}$ a subfunctor of the functor of definable types $S_*^{def} : \mathrm{Mod}(T) \to \mathrm{Set}$ (which satisfies some natural closure properties). We fix a monster model $\mathbb{U}$ of T and all structures to us later will be small substructures of $\mathbb{U}$. We use $\mathcal{L}_P$ to denote $\mathcal{L}$ expanded by a new predicate P. And we write $A \downarrow_C B$ if $tp_{\mathcal{L}}(A/BC)$ admits a global extension that is C-definable.

We say a $\mathcal{L}_P$-structure $(A, P(A))$ is *pairial* if $tp_{\mathcal{L}}(A/P(A))$ admits a global extension in $\mathcal{C}$ that is $P(A)$-definable. And an $\mathcal{L}_P$-embedding $f : A \to B$ is *pairial* if $f(A) \downarrow_{f(P(A))} P(B)$.

And we will be interested in the rich (possibly large) structures in the category of pairial structures $\mathcal{F}_{\mathcal{C}}$.

Example

In ACF, any pairs of fields (K, L) is pairial.
In DOAG, $(\mathbb{R}, \mathbb{Q})$ is not pairial, but, $(\mathbb{R}(\epsilon), \mathbb{R})$ is pairial.

Definition

A (large) $\mathcal{L}_P$-structure $(M, P(M))$ is called a $\mathcal{C}$-*beautiful pair* if the following are satisfied

(i) $P(M) \prec M \models T$ and $tp_{\mathcal{L}}(M/P(M)) \in \mathcal{C}$

(ii) Whenever there are pairial embeddings $f\colon (A, P(A)) \to (M, P(M))$ and $g\colon (A, P(A)) \to (B, P(B))$, there is a pairial embedding $h\colon (B, P(B)) \to (M, P(M))$ such that $f(a) = h(g(a))$ for all $a \in A$.

Pairal structures and $\mathcal{C}$-beautiful pairs

Theorem

$\mathcal{F}_\mathcal{C}$ satisfies the following

1. *(Extension property) If $(A, P(A))$ is in $\mathcal{F}_\mathcal{C}$ and $\varphi(x)$ is a consistent $\mathcal{L}(A)$-formula, then there are a pairial embedding $f : (A, P(A)) \to (B, P(B))$ and $b \in B$ satisfying $\varphi(x)$.*
2. *(Amalgamation property) Given pairial embeddings $f_1 : (A, P(A)) \to (B_1, P(B_1))$ and $f_2 : (A, P(A)) \to (B_2, P(B_2))$ between elements in $\mathcal{C}$, there is $(C, P(C))$ in $\mathcal{F}_\mathcal{C}$ and pairial embeddings $g_i : (B_i, P(B_i)) \to (C, P(C))$ for $(i = 1, 2)$ such that $g_1 \circ f_1(a) = g_2 \circ f_2(a)$ for all $a \in A$.*

iff $\mathcal{C}$-beautiful pairs exist.
Moreover, all $\mathcal{C}$-beautiful pairs are elementarily equivalent.

Axiomtizing $\mathcal{C}$-beautiful pairs

When $T = \mathrm{ACF}$ (or and $\mathcal{C}$ is exactly the definable types, the beautiful pairs will be exactly the ones given by Poizat. Moreover, in this case, any sufficiently saturated model of their common theory is a beautiful pair. In particular, the predicate is pure. And one gets quantifier elimination by adding the functions f_φ that assigns each $a \in N$ to the canonical parameter of the set $\{c \in M : \varphi(x, c) \in tp_{\mathcal{L}}(a/M)\}$.
Similarly, in the case of real closed fields and $\mathcal{C}$ is the functor of definable types, the common theory of $\mathcal{C}$-beautiful pairs is exactly T_{tame} as described by van den Dries and Lewenberg.

Axiomitizing $\mathcal{C}$-beautiful pairs

In DOAG, take $\mathcal{C}$ to be all bounded definable types. Extension property and amalgamations are clear in this case.
Moreover, we have an axiomitization of it.

(i) $M \prec N \models \mathrm{DOAG}$ and M is Dedekind complete in N.

(ii) M is cofinal in N

and saturated models of the above are themselves $\mathcal{C}$-beautiful pairs.
Modifying (ii), one gets similar statements for $C = S^{def}$ and types concentrating at infinity as well.

Beautiful pairs of valued fields

It is well known that for pairs of (nice) valued fields, (L, K) is pairial implies that L/K is *separated*, i.e. any n-dimensional K-vector space V in L admits a K-basis $\{v_1, ..., v_n\}$ such that $\mathrm{val}(c_1 v_1 + ... + c_n v_n) = \min \mathrm{val}(c_i v_i)$.

Theorem

Let T be a "nice" theory of valued fields, the theory of $\mathcal{C}$-beautiful pairs (L, K) can be axiomtized by the following:

- $K \preceq L \models T$;
- $K \subseteq L$ *is separated;*
- $\mathrm{RES}(K) \preceq \mathrm{RES}(L)$ *and* $\Gamma(K) \preceq \Gamma(L)$ *are models of the common theory of their corresponding beautiful pairs.*

Moreover, sufficiently saturated models of such theories are $\mathcal{C}$-beautiful pairs.

Beautiful pairs of valued fields

Specific examples of such nice valued fields includes :

- Algebraically closed valued fields
- Real closed valued fields
- $\mathbb{C}((t))$ $\mathbb{R}((t))$
- $\mathbb{Q}_p$

In the context of ACVF, we have the following correspondence.

$\mathcal{C}$	RES	Γ	Spaces
S_V^{def}	$k \prec l$	$\Gamma(K) \prec_{full} \Gamma(L)$	Zariski-Riemann spaces
$\widehat{V}$	$k \prec l$	$\Gamma(K) = \Gamma(L)$	Berkovich spaces
$\widetilde{V}$	$k \prec l$	$\Gamma(K) \prec_{bdd} \Gamma(L)$	Adic spaces
Infinitesimal types	$k = l$	$\Gamma(K) \prec_{\infty} \Gamma(L)$	???

From now on, we will focus on ACVF.

Definition

Let p be a definable type, we say that p is *bounded* if for some model K over which p is defined, there is $K \preceq L$ with a realization of $p|K$ in L such that $\Gamma(K)$ is cofinal in $\Gamma(L)$.

We use $\widetilde{V}$ to denote the set of bounded definable types on V, and $\widetilde{V}(K)$ to denote those definable over K. We use $\widehat{V}$ to denote the set of generically stable types on V.

Theorem

$\widetilde{V}, \widehat{V}$ are strict pro-definable.

Connection to adic/Berkovich spaces

Model theoretically, given $K \models$ ACVF and K complete with respect to the valuation, with value group $\mathbb{R}$. Let V be an variety over K, we have

$$V^{an} = \{p \in S_V(K) : \text{p is weakly orthogonal to } \Gamma\}$$
$$V^{ad} = \{p \in S_V(K) : \text{p is bounded in } \Gamma(K)\}$$
$$\mathrm{RZ}(V) = S_V(K)$$

For any such K, recall that one can find a speherically complete ACVF K^{max} extending K with value group $\mathbb{R}$. One can define the restriction map to descend various maps on the definable level.

$$\pi : \widehat{V}(K^{max}) \to V^{an}$$
$$\pi : \widetilde{V}(K^{max}) \to V^{ad}$$
$$\pi : S_V^{def}(K^{max}) \to \mathrm{RZ}(V)$$

An important feature in the Hrushovski-Loeser theory is the canonical extension. Namely, to define a map $h : \widehat{V} \to \widehat{W}$, it suffices to define a map $h' : V \to \widehat{W}$, there is a canonical extension to $\widehat{V}$. Similar feature exists in the category of $\widetilde{V}$, and the two extensions are compatible with the restriction given by $\widehat{V} \subseteq \widetilde{V}$. And in particular, we hope that the lifts of nice continuous functions should be continuous.

Like the adic space, one can topologize $\widetilde{V}$ with two different topologies.

Definition

Let U be a Zariski open and $f, g \in \mathcal{O}_V(U)$. Topologize $\widetilde{V}$ by the weakest topology such that $\{p : \infty \neq v \circ f_*(p) \leq v \circ g_*(p)\}$ is open. We topologize $\widehat{V}$ by the sets of the form $\{p \in \widehat{V} : v \circ f_*(p) < v \circ g_*(p)\}$. We say an open subset of $\widetilde{V}$ is *partially proper* if it is closed under specialization and use $\widetilde{V}_{p.p}$ to denote $\widetilde{V}$ with the partially proper topology.

Example

The valuation ring $\widetilde{\mathcal{O}}$ is an open subset of $\widetilde{\mathbb{A}^1}$ but not partially proper since the ball with valuative radius 0^- is the specialization of the generic type of $\mathcal{O}$. However, let $\mathfrak{m}$ denote the maximal ideal of $\mathcal{O}$. Consider $U = \widetilde{\mathfrak{m}} \backslash \{p\}$ where p is the generic type of $\mathfrak{m}$, one can check that U is partially proper open.

Note that in the above example. Consider $\widehat{V} \subseteq \widetilde{V}$, $U \cap \widehat{\mathbb{A}^1} = \widehat{\mathfrak{m}}$, which is an open subset of $\widehat{\mathbb{A}^1}$.

Canonical extensions

The above follows from a general fact, as in the comparison of Berkovich and Huber's analytification.

Theorem

$\widehat{V} \subseteq \widetilde{V}$ and the topology on $\widehat{V}$ is the induced topology of $\widetilde{V}_{p.p}$.

Canonical extensions

Theorem (Hrushovski,Loeser)

Let V be a quasi-projective variety, there is a pro-definable deformation retraction $h : I \times \widehat{V} \to \widehat{V}$ with a Γ-internal and iso-definable image.

Theorem (Cubides Kovacsics, Y.)

The above deformation retraction lifts to $H : I \times \widetilde{V}_{p.p} \to \widetilde{V}_{p.p}$.

Thank you for your attention!