Adding a multiplicative group to a polynomially bounded structure.

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1. Results

$\overline{\mathbb{R}} :=$ $\langle \mathbb{R}; +, \cdot, -, 0, 1, \dots, < \rangle$ the ordered field of real numbers (possibly with extra constants).

$\tilde{\mathbb{R}} :=$ $\langle \overline{\mathbb{R}}; \dots \rangle$, some expansion of $\overline{\mathbb{R}}$.

$\tilde{T} :=$ $Th(\tilde{\mathbb{R}})$, the theory of $\tilde{\mathbb{R}}$.

Assume that

(1.1) $\tilde{T}$ is o-minimal;

(1.2) $\tilde{T}$ is polynomially bounded with field of exponents $\mathbb{Q}$;

(1.3) $\tilde{T}$ is model complete.

1.4 Main example

- For each $n \geq 0$, open $U \subseteq \mathbb{R}^n$, analytic function $f : U \to \mathbb{R}$, and closed, bounded box $B \subseteq U$, define $f^{\upharpoonright} : \mathbb{R}^n \to \mathbb{R}$ by

$$f^{\upharpoonright}(\bar{x}) := \begin{cases} f(\bar{x}) & \text{if } \bar{x} \in B; \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathbb{R}_{an} := \langle \bar{\mathbb{R}}, \{f^{\upharpoonright}\}_f \rangle$ satisfies (1.1), (1.2) and (1.3). (Gabrielov, Denef-van den Dries.)

- Obviously any reduct of $\mathbb{R}_{an}$ also satisfies (1.1) and (1.2). Gabrielov showed that if we take any subcollection $\mathcal{S} \subseteq \{f^{\upharpoonright}\}_f$ which is closed under differentiation, then

(Gab.) $\mathbb{R}_{\mathcal{S}} := \langle \bar{\mathbb{R}}, \mathcal{S} \rangle$

also satisfies (1.3)

1.5 Adding a multiplicative group.

Let $G(\cdot)$ be a new unary predicate symbol. Let ω be a constant symbol of $\mathcal{L}(\tilde{T})$ such that (in $\tilde{\mathbb{R}}$) $\omega > 1$.

Consider the following axioms DMG in the language $\mathcal{L}(\tilde{T}) \cup \{G\}$:-

DMG(1) $\quad \forall x \, (G(x) \rightarrow x > 0)$;

DMG(2) $\quad \forall x, y \, ((G(x) \wedge G(y)) \rightarrow G(x \cdot y))$;

DMG(3) $\quad \forall x, y \, ((G(x) \wedge x \cdot y = 1) \rightarrow G(y))$;

DMG(4) $\quad \forall x > 1 \, (G(x) \rightarrow x \geqslant \omega)$;

DMG(5) $\quad \forall x > 0 \, \exists y \, (G(y) \wedge y \leq x < \omega \cdot y)$.

necessarily unique.

1.6 Main Theorem

$\tilde{T} \cup$ DMG is complete and model complete.

1.7 Remarks

(a) Clearly the only expansion of $\tilde{\mathbb{R}}$ to a model of $\tilde{T} \cup DMG$ is

$$\langle \tilde{\mathbb{R}} ; \omega^{\mathbb{Z}} \rangle .$$

Further, if $\tilde{M} \preccurlyeq \tilde{\mathbb{R}}$ is the minimal model of $\tilde{T}$, then $\omega^{\mathbb{Z}} \subseteq M$ and $\langle \tilde{M} ; \omega^{\mathbb{Z}} \rangle$ $(\models \tilde{T} \cup DMG)$ is embeddable in every model of $\tilde{T} \cup DMG$. So completeness follows from model completeness.

[Our results all go through for theories $\tilde{T}$ satisfying (1.1), (1.2) and (1.3) without the assumption that the minimal model of $\tilde{T}$ is archimedean. One must then specify in DMG which 0-definable elements satisfy $G(x)$.]

(b) The case $\tilde{\mathbb{R}} = \mathbb{R}$ of the Main Theorem was proved in 1985 by van den Dries. Our proof is the same! but has surprising consequences....

1.8 Some consequences

(a) (Some model theory for real analytic, periodic functions.)

Let $\bar{\mathbb{R}} = \langle \mathbb{R}; +, \cdot, -, 0, 1, e^{2\pi}, < \rangle$.

Let $\mathcal{F}$ be any collection of analytic, 2π-periodic functions (from $\mathbb{R}$ to $\mathbb{R}$) closed under differentiation.

Assume $\sin, \cos \in \mathcal{F}$.

For $f \in \mathcal{F}$, define $f^* : \mathbb{R} \to \mathbb{R}$ by

$$f^*(x) = \begin{cases} f(\log x), & x > 0 \\ 0, & x \leq 0. \end{cases}$$

Let $G = e^{2\pi\mathbb{Z}}$, and note that

$$\forall x \in \mathbb{R},\ \forall g \in G,\ f^*(g \cdot x) = f^*(x) \ \ldots\ (\text{MPer.})$$

1.8.1 Corollary

$Th(\langle \bar{\mathbb{R}}; \{f^* : f \in \mathcal{F}\} \rangle)$ is model complete.

Proof.

Let $\mathcal{S}'$ be the collection of all functions of the form $(0,\infty) \to \mathbb{R} : x \longmapsto P(x^s, f_1^*(x), \dots, f_r^*(x))$ for P a polynomial (over $\mathbb{Z}$), $s \in \mathbb{Z}$ and $f_1, \dots, f_r \in \mathcal{F}$. Let $\mathcal{S} = \{ F \restriction [1/N, N] : N \geq 1, F \in \mathcal{S}' \}$. Then $\mathcal{S}$ is closed under differentiation, so $\langle \bar{\mathbb{R}}, \mathcal{S} \rangle$ is a structure of type (Gab.), and hence the Main Theorem applies. So $Th(\langle \bar{\mathbb{R}}, \mathcal{S}, G \rangle)$ is model complete. But clearly, using (MPer), the structures $\langle \bar{\mathbb{R}}; \mathcal{S}, G \rangle$ and $\langle \bar{\mathbb{R}}; \{ f^* : f \in \mathcal{F} \} \rangle$ are $\exists \cap \forall$-bi-interpretable. (NB $G = \{ x > 0 : \sin \log x = 0, \cos \log x = 1 \}$.)

(b) Let $\tilde{\mathbb{C}}$ be the expansion of the complex field by the (many valued) function $z \longmapsto z^i$, i.e. by the relation $g := \{ \langle e^u, e^{iu} \rangle \in \mathbb{C}^2 : u \in \mathbb{C} \}$.

Note that $\forall z, w \in \mathbb{C}$, $\forall g, h \in G$ $(= e^{2\pi \mathbb{Z}})$,

$$\langle z, w \rangle \in \mathcal{G} \implies \langle gz, hw \rangle \in \mathcal{G}.$$

Using this, it is easy to find a structure $\hat{\mathbb{R}}$ of type (fab) such that $\tilde{\mathbb{C}}$ is $\forall \wedge \exists$-interpretable in $\langle \tilde{\hat{\mathbb{R}}}, G \rangle$, (via $\mathbb{C} \sim \mathbb{R}^2$).

- Known methods + Main Theorem (and its proof) now yield:

1.8.2 <u>Corollary</u>

$\tilde{\mathbb{C}}$ is quasi-minimal (i.e. every definable subset of $\mathbb{C}$ is either countable or co-countable).

(I think $\tilde{\mathbb{C}}$ is the first reduct of $\langle \mathbb{C}, +, \cdot, 0, 1, \exp \rangle$, properly expanding $\langle \mathbb{C}, +, \cdot, 0, 1 \rangle$, for which this is known.)

2. o-minimal structures

$\overline{M} = \langle M; +, \cdot, -, 0, 1, < \rangle$ an RCOF.

$\tilde{M} = \langle \overline{M}, \ldots \rangle$ an expansion of $\overline{M}$.

$\tilde{M}$ is <u>o-minimal</u> if every definable subset of M is a finite union of open intervals and points. Let $\tilde{T} = Th(\tilde{M})$.

<u>Facts</u>

2.1 o-minimality implies many topological and geometrical finiteness conditions on arbitrary definable $S \subseteq M^n$. (Pillay/Steinhorn/Knight; see van den Dries' book.)

2.2 $\tilde{N} \equiv \tilde{M} \Rightarrow \tilde{N}$ o-minimal (P/S).

2.3 $\tilde{T}$ is a <u>Skolem</u> theory and Skolem closure is a pregeometry:

Let $\tilde{M}_1, \tilde{M}_2 \models \tilde{T}$, $\tilde{M}_1 \preccurlyeq \tilde{M}_2$.

Let $S \subseteq M_2$.

Then the closure of $M_1 \cup S$ under the 0-definable functions (of $\tilde{M}_2$), denoted $M_1(S)$, is the domain of a (unique) elementary substructure, $\widetilde{M_1(S)}$, of $\tilde{M}_2$:

$$\tilde{M}_1 \preccurlyeq \widetilde{M_1(S)} \preccurlyeq \tilde{M}_2 .$$

- Further, $\exists\ S_0 \subseteq S$ such that $M_1(S_0) = M_1(S)$ (i.e. S_0 generates $M_1(S)$ over M_1) and $\forall s \in S_0$ $s \notin M_1(S_0 \setminus \{s\})$ (i.e. S_0 is independent over M_1).
- All such bases (over M_1) have the same cardinality, denoted $\dim_{\tilde{M}_1}(\widetilde{M_1(S)})$.

3. Polynomially bounded structures

An o-minimal structure $\tilde{M}$ is called polynomially bounded (with $\mathbb{Q}$ exponents), if for all definable $f : M \to M$, $\exists\ q \in \mathbb{Q}$, $\exists\ c \in M$ such that $\dfrac{f(x)}{c \cdot x^q} \to 1$ as $x \to +\infty$ (in $\tilde{M}$).

- Holds for Main Example. False for $\langle \bar{\mathbb{R}}, \exp \rangle$, $\langle \bar{\mathbb{R}}, x \mapsto x^{\sqrt{2}} \rangle$.

Now assume $\tilde{T} = Th(\tilde{\mathbb{R}})$ with $\tilde{\mathbb{R}}$ as in section 1. (Only (1.1), (1.2) are needed here.)

For $\tilde{M} \models \tilde{T}$ let $\Gamma(\tilde{M})$ be the (multiplicative) group of skies of $\langle M^{>0}, \cdot \rangle$. A sky is an equivalence class of the relation:

$$(\text{for } a, b \in M^{>0}) \quad a \sim b \iff \exists N \in \mathbb{N} \setminus \{0\}, \ \frac{1}{N} \leq \frac{a}{b} \leq N.$$

$\Gamma(\tilde{M})$ inherits multiplication from $\tilde{M}$ and is a divisible abelian group (since $\langle M^{>0}, \cdot \rangle$ admits n^{th}-roots $\forall n$), hence a $\mathbb{Q}$-vector space (written multiplicatively!).

3.1 The Valuation Inequality (Wilkie, v.d.Dries/ Speissegger).

(a) Let $\tilde{M}_1, \tilde{M}_2 \models \tilde{T}$, $\tilde{M}_1 \leq \tilde{M}_2$ and suppose that $\dim_{\tilde{M}_1}(\tilde{M}_2) = d$ $(< \infty)$. Then

$$\dim_{\mathbb{Q}} \frac{\Gamma(\tilde{M}_2)}{\Gamma(\tilde{M}_1)} \leq d.$$

(b) In particular (for $d = 1$), if $\tilde{M}_2 = \widetilde{M_1(a)}$

for some $a \in M_2$, and if there is a new sky, $g/\sim$ say, then

$$\forall \beta \in M_2 \setminus \{0\} \quad \exists \alpha \in M_1 \setminus \{0\},\ \exists q \in \mathbb{Q}$$

such that $\frac{1}{N} \leq \left| \frac{\beta}{\alpha \cdot g^q} \right| \leq N$ for some $N \in \mathbb{N} \setminus \{0\}$.

4. Proof of the Main Theorem

$\tilde{T} = Th(\tilde{\mathbb{R}})$ as in (1.1), (1.2) and (1.3).

Our (and van den Dries') proof is based on the following:

4.1 Lemma

Suppose that $\langle \hat{\tilde{M}}_1, G_1 \rangle$, $\langle \hat{\tilde{M}}_2, G_2 \rangle$ are both models of $\tilde{T} \cup DMG$ with $\langle \hat{\tilde{M}}_1, G_1 \rangle \subseteq \langle \hat{\tilde{M}}_2, G_2 \rangle$ (so that $\hat{\tilde{M}}_1 \underset{\mathcal{L}(\tilde{T})}{\preceq} \hat{\tilde{M}}_2$ by (1.3)). Let S be any subset of G_2. Then $\langle \widehat{M_1(S)}, M_1(S) \cap G_2 \rangle \models \tilde{T} \cup DMG$.

Proof.

By induction, we may suppose that $S = \{g\}$ for some $g \in G_2$.

Clearly we only have to show that

$$\langle \widetilde{M_1(S)},\ M_1(S) \cap G_2 \rangle \models \mathrm{DMG}(5) \quad \text{(as}$$

all the other axioms are inherited from $\widetilde{M}_2$.)

So let $a \in M_1(g)$, $a > 0$.

$$\left[\begin{array}{l} \widetilde{M}_1 \leq \widetilde{M_1(g)} \leq \widetilde{M}_2 \\ \omega \in G_1 \subseteq G_2 \cap M_1(g) \subseteq G_2 \end{array}\right]$$

Since $\langle \widetilde{M}_2, G_2 \rangle \models \mathrm{DMG}(5)$, there is some $h \in G_2$ such that $h \leq a < \omega \cdot h$.

We shall show that $h \in M_1(g)$.

Obviously we may assume that $g \notin M_1$ (otherwise $M_1 = M_1(g)$ and lemma is trivial).

It follows that $g/_n$ is a new sky (i.e. not represented in $\widetilde{M}_1$) because otherwise $\exists\, b \in M_1$, $\exists N \in \mathbb{N} \setminus \{0\}$ such that

$$\frac{1}{N} \leq g/_b \leq N$$

and by DMG(5) in $\widehat{M}_1$ we may assume that $b \in G_1$. But then

$$g/b = \omega^m \quad (\text{some } m \in \mathbb{Z})$$

so $\quad g = b \cdot \omega^m \in M_1$, a contradiction

Hence, by the Valuation Inequality (3.1(b)), there is some $e \in M_1$ and $q \in \mathbb{Q}$, and $N \in \mathbb{N} \setminus \{0\}$ s.th.

$$\frac{1}{N} \leq \frac{a}{e \cdot g^q} \leq N.$$

As above, we may suppose $e \in G_1$ and we may also replace a by h. (NB $N' \geq \omega \geq \frac{1}{N'}$ for some $N' \in \mathbb{N} \setminus \{0\}$).

Also, if $q = s/t$ $\quad (s, t \in \mathbb{Z}, t \neq 0)$, then

$$\frac{1}{N^t} \leq \underset{\substack{\cap \\ G_2}}{\frac{h^t}{e^t g^s}} \leq N^t \quad \text{and}$$

So $$\frac{h^t}{e^t \cdot g^s} = \omega^m \quad (\text{some } m \in \mathbb{Z}).$$

Thus $$h^t = \omega^m \cdot e^t \cdot g^s \in M_1(g).$$

But $\widetilde{M_1(g)}$ is a real closed field,

$\therefore$ $h \in M_1(g)$ as required. $\square$

<u>Proof of ~~Main~~ Theorem</u>

By Robinson's Test, we must show the following (using (1.3)):

Suppose $\langle \tilde{M}_1, G_1 \rangle, \langle \tilde{M}_2, G_2 \rangle \models \tilde{T} \cup DMG$, $\tilde{M}_1 \leq \tilde{M}_2$, $G_1 = G_2 \cap M_1$ and $g_1, \ldots, g_n \in G_2$. Then if $\phi(x_1, \ldots, x_n)$ is a formula of $\mathcal{L}(\tilde{T})$ with parameters in M_1, and if $\tilde{M}_2 \models \phi[g_1, \ldots, g_n]$, then for some $h_1, \ldots, h_n \in G_1$, $\tilde{M}_1 \models \phi[h_1, \ldots, h_n]$.

Now if $n \geq 2$, we may argue by induction, by replacing $\tilde{M}_1$ with $\tilde{M}_1(\{g_1, \ldots, g_{n-1}\})$ and using Lemma 4.1, provided we can do the case $n = 1$.

But since $\phi(x, \,)$ defines a finite union of open intervals and points – with all endpoints in M_1 – this follows easily from the axioms DMG.

$\square$