Hensel minimality and counting in valued fields

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O-minimality and non-Archimedean analogues

O-minimality

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If the language $\mathcal{L}$ expands $\mathcal{L}_{\mathrm{oring}} = \{0, 1, +, \cdot, <\}$ and M is an o-minimal ordered ring, then M is a real closed field.

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4. Dimension theory.

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3. b-minimality (Cluckers–Loeser).
4. V-minimality (Hrushovski–Kazhdan).

Hensel minimality

An alternative definition of o-minimality

Let $\mathcal{L}$ expand $\mathcal{L}_{\mathrm{oring}}$. Then R is o-minimal if for every definable $X \subset R$ there exists a finite set C such that the following holds: if $x, y \in R$ are such that

$$\forall c \in C : \mathrm{sgn}(x - c) = \mathrm{sgn}(y - c),$$

then either x, y are both in X, or they are both not in X.

Notation and the leading term structures

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For $\lambda \leq 1$ in $\Gamma_K^\times$ define

$$\mathrm{RV}_\lambda^\times = \frac{K^\times}{1 + B_{<\lambda}(0)}$$

to be the *leading term structure of order* λ. Let $\mathsf{rv}_\lambda : K \to \mathrm{RV}_\lambda$ be the natural map ($\mathsf{rv}_\lambda(0) = 0$). Let $\mathrm{RV} = \mathrm{RV}_1, \mathsf{rv} = \mathsf{rv}_1$.

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$$1 \to k^\times \to \mathrm{RV}^\times \to \Gamma_K^\times \to 1.$$

Balls λ-next and preparation

Let $\lambda \leq 1$ in $\Gamma_K^\times$, $c \in K$ and $\xi \in \mathrm{RV}_\lambda$. A *ball λ-next to c* is a ball of the form

$$\{x \in K \mid \mathrm{rv}_\lambda(x-c) = \xi\} = c + \mathrm{rv}_\lambda^{-1}(\xi).$$

This is an open ball of radius $|\xi|$.

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If C is a finite set in K then a ball λ-next to C is a (non-empty) set $\cap_{c \in C} B_c$ where every B_c is λ-next to c.

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A set $X \subset K$ is *λ-prepared* by a finite set $C \subset K$ if every ball λ-next to C is either contained in X or disjoint from X.

Naive definition

A field K is h-minimal if for every definable $X \subset K$ there exists a finite definable set C such that the following holds: if $x, y \in K$ are such that

$$\forall c \in C : \mathrm{rv}(x - c) = \mathrm{rv}(y - c),$$

then either x, y are both in X, or they are both not in X.

Precise definition

Let $\mathcal{L}$ expand $\mathcal{L}_{\mathrm{val}} = \mathcal{L}_{\mathrm{ring}} \cup \{\mathcal{O}\}$ and let $\mathcal{T}$ be a complete $\mathcal{L}$-theory containing $\mathcal{T}_{\mathrm{val}_{0,0}}$. Let $\ell \in \mathbb{Z}_{\geq 0} \cup \{\omega\}$. Then $\mathcal{T}$ is ℓ-h-minimal if for every model K of $\mathcal{T}$ the following holds:

> For every $\lambda \leq 1$ in $\Gamma_K^\times$, for every $A \subset K$ and every $A' \subset \mathrm{RV}_\lambda$ of cardinality $\#A' \leq \ell$, every $(A \cup A' \cup \mathrm{RV})$-definable set $X \subset K$ can be λ-prepared by a finite A-definable set $C \subset K$.

Henselianity

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Let K be a valued field of equicharacteristic 0. Then the following are equivalent.

1. *K is Henselian,*
2. $\text{Th}_{\mathcal{L}_{\text{val}}}(K)$ *is 0-h-minimal,*
3. $\text{Th}_{\mathcal{L}_{\text{val}}}(K)$ *is ω-h-minimal.*

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3. $\mathbb{R}((t))$ in $\mathcal{L}_{\mathrm{val}} \cup \{<\}$.
4. More generally, ℓ-h-minimality is preserved under RV-expansion.

Non-examples

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If K is 0-h-minimal in $\mathcal{L}$, then we cannot have a section $k \to K$ or $\Gamma_K \to K$ in the language. Neither can we have an automorphism of K in the language. Also, every definable map $\mathrm{RV}^n \to K$ has finite image.

Mixed characteristic

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> For every $\lambda \leq 1$ in $\Gamma_K^\times$, for every $A \subset K$ and every $A' \subset \mathrm{RV}_\lambda$ of cardinality $\#A' \leq \ell$, for every $(A \cup A' \cup \mathrm{RV})$-definable set $X \subset K$ there exists a positive integer m and a finite A-definable set $C \subset K$ such that C $|m|\lambda$-prepares X.

Mixed characteristic examples

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1. Any Henselian valued field of mixed characteristic in $\mathcal{L}_{\mathrm{val}}$. In particular $\mathbb{Q}_p$.

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1. Any Henselian valued field of mixed characteristic in $\mathcal{L}_{\mathrm{val}}$. In particular $\mathbb{Q}_p$.
2. $\mathbb{Q}_p$ with the subanalytic language.

Tameness under h-minimality

The Jacobian property

From now on, let K be a valued field of equicharacteristic 0, equipped with a 1-h-minimal $\mathcal{L}$-structure.

Theorem (Jacobian property)

Let $f : K \to K$ be a definable function. Then there exists a finite definable set C such that for every ball B 1-next to C there exists a $\xi_B \in \mathrm{RV}$ such that if $x, y \in B$ are distinct, then

$$\mathrm{rv}\left(\frac{f(x) - f(y)}{x - y}\right) = \xi_B.$$

Differentiability

Theorem (Jacobian property)

Let $f : K \to K$ be a definable function. Then there exists a finite definable set C such that for every ball B 1-next to C, f is differentiable, $\mathrm{rv} \circ f'$ is constant on B and for $x, y \in B$ distinct we have

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If $B' \subset B$ is an open ball, then $f(B')$ is an open ball of radius $|f'(x)| \,\mathrm{rad}_{\mathrm{op}}\, B'$.

Taylor approximation

Theorem (Taylor approximation)

Let $f : K \to K$ be definable and let $r \in \mathbb{N}$. Then there exists a finite definable set $C \subset K$ such that for every ball B 1-next to C, $f^{(r+1)}$ exists, $|f^{(r+1)}|$ is constant on B and for all $x_0, x \in B$ we have

$$|f(x) - T^{\leq r}_{f,x_0}(x)| \leq |f^{(r+1)}(x_0)| \cdot |x - x_0|^{r+1}.$$

Algebraic Skolem functions

A theory $\mathcal{T}$ has algebraic Skolem functions if for every model K of $\mathcal{T}$ and every $A \subset K$ we have $\mathrm{acl}_K(A) = \mathrm{dcl}_K(A)$.

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If $\mathrm{Th}_{\mathcal{L}}(K)$ is ℓ-h-minimal then there exists an expansion $\mathcal{L}' \supset \mathcal{L}$ such that $\mathrm{Th}_{\mathcal{L}'}(K)$ is ℓ-h-minimal and has algebraic Skolem functions.

Cells in K^2

Let $R \subset \mathrm{RV}^2$, a (definable) cell X in K^2 consists of $(x, y) \in K^2$ for which

$$(\mathrm{rv}(x - c_1), \mathrm{rv}(y - c_2(x))) \in R,$$

with c_1 definable, and $c_2 : K \to K$ a definable function.

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with c_1 definable, and $c_2 : K \to K$ a definable function. Let

$$(RV^\times)^1 = \mathrm{RV}^\times, (\mathrm{RV}^\times)^0 = \{0\}.$$

If $R \subset (\mathrm{RV}^\times)^{j_1} \times (\mathrm{RV}^\times)^{j_2}$ then X is a (j_1, j_2)-cell.

Cell decomposition

Assume that $\mathrm{Th}_{\mathcal{L}}(K)$ has algebraic Skolem functions and that it is 1-h-minimal.

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Let $X \subset K^n$ be definable. Then there exists a partition of X into finitely many definable cells.

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Theorem (Cell decomposition II)

Let $f : K^n \to K$ be definable. Then there exists a definable cell decomposition of K^n such that f is continuous on each cell of this cell decomposition.

Dimension theory

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1. $\dim X = 0$ *iff* X *is finite.*
2. $\dim(X \cup Y) = \max\{\dim X, \dim Y\}$.
3. *The set of* $z \in Z$ *for which* $\dim f^{-1}(z) = d$ *is definable (for any* d*).*
4. *If* $\dim f^{-1}(z) = d$ *for all* $z \in Z$ *then* $\dim X = d + \dim Z$.
5. *We have* $\dim(\overline{X} \setminus X) < \dim X$.

Theorem

Let $f : K^n \to K$ be definable. Then the set U of $u \in K^n$ such that f is C^k on a neighbourhood of u is dense in K^n.

Supremum Jacobian property

For $\lambda, \mu \in \Gamma_K$ we define $\lambda <_0 \mu$ if $\lambda < \mu$ or $\lambda = \mu = 0$. Then $\text{rv}(a) = \text{rv}(b)$ iff $|a - b| <_0 |a|$.

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Let $f : K \to K$ definable. The Jacobian property says

$$\text{rv}(f(x) - f(y)) = \text{rv}(f'(x) \cdot (x - y)).$$

Equivalently

$$|f(x) - f(y) - f'(x) \cdot (x - y)| <_0 |f'(x)||x - y|.$$

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Let $f : K^n \to K$ definable. The supremum Jacobian property says that there exists a definable map $\chi : K^n \to \mathrm{RV}^m$ such that

$$|f(x) - f(y) - (\mathrm{grad}\, f(y)) \cdot (x - y)| <_0 |\mathrm{grad}(f(y))||x - y|$$

on the n-dimensional fibres of χ.

Counting in valued fields

Heights and counting

For $a/b \in \mathbb{Q}$, $\gcd(a, b) = 1$, let $H(a/b) = \max\{|a|, |b|\}$ be the height. For $x = (x_1, \ldots, x_n) \in \mathbb{Q}^n$ let $H(x) = \max_i H(x_i)$. For $X \subset \mathbb{R}^n$ or $X \subset \mathbb{Q}_p^n$, let

$$X(\mathbb{Q}, B) = \{x \in X \cap \mathbb{Q}^n \mid H(x) \leq B\},$$
$$X(\mathbb{Z}, B) = \{x \in X \cap \mathbb{Z}^n \mid H(x) \leq B\}.$$

The Bombieri–Pila theorem

Theorem (Bombieri–Pila)

Let $X \subset \mathbb{R}^2$ be an irreducible algebraic curve of degree d. Then for every $\varepsilon > 0$ there exists $c_{d,\varepsilon} > 0$ such that for all B

$$\#X(\mathbb{Z}, B) \leq c_{d,\varepsilon} B^{1/d+\varepsilon}$$

The Pila–Wilkie theorem

For $X \subset \mathbb{R}^n$ let X^{alg} be the union of all semialgebraic connected subsets of X of dimension 1. Let $X^{\mathrm{trans}} = X \setminus X^{\mathrm{alg}}$ be the *transcendental part of* X.

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Theorem (Pila–Wilkie)

Let $X \subset \mathbb{R}^n$ be definable in an o-minimal structure. Then for any $\varepsilon > 0$ there is $c_\varepsilon > 0$ such that for all B

$$\# X^{\mathrm{trans}}(\mathbb{Q}, B) \leq c_\varepsilon B^\varepsilon.$$

A Pila–Wilkie theorem for $\mathbb{Q}_p$

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Theorem (Cluckers–Comte–Loeser, Cluckers–Forey–Loeser)

Let $X \subset \mathbb{Q}_p^n$ be $\mathcal{L}$-definable. Then for every $\varepsilon > 0$ there exists a $c_\varepsilon > 0$ such that for all B

$$\#X^{\mathrm{trans}}(\mathbb{Q}, B) \leq c_\varepsilon B^\varepsilon.$$

A Bombieri–Pila theorem for $\mathbb{C}((t))$

Let $\mathbb{C}[t]_{<s}$ be all polynomials of degree $< s$. For $X \subset \mathbb{C}((t))^n$ put $X_s = X \cap \mathbb{C}[t]^n_{<s}$. Let $n_s(X)$ be the Zariski dimension of the Zariski closure of $X_s \subset \mathbb{C}[t]^n_{<s} = \mathbb{C}^{ns}$.

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Theorem (Cluckers–Comte–Loeser)

Let $X \subset \mathbb{C}((t))^2$ be an irreducible algebraic curve of degree d. Then for every s

$$n_s(X) \leq \left\lceil \frac{s}{d} \right\rceil .$$

Rational points

For $K = \mathbb{Q}_p$ and $X \subset K^n$ let $X(\mathbb{Q}, B)$ and $X(\mathbb{Z}, B)$ be as defined above and put $X_s = X(\mathbb{Z}, p^s)$. These are *rational points of height at most* B.

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For $K = k((t))$ let $k[t]_{<s}$ be all polynomials of degree $< s$. For $X \subset K^n$ define $X_s = X \cap k[t]^n_{<s}$. These are *rational points of height at most* s. How to measure the size of X_s as $s \to \infty$?

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Possible generalizations: other heights on $\mathbb{Q}_p$, Teichmüller lifts; other fields with a (pseudo-)uniformizer, localizations ($k[t, t^{-1}]$), multiple uniformizers, ...

The counting dimension

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Let $K = k((t))$ or $K = \mathbb{Q}_p$ and let $\mathcal{L}$ be a language extending $\mathcal{L}_{\mathrm{val}}$.

Definition

Let $X \subset K^n$, let d be a positive integer and let $N, e : \mathbb{N} \to \mathbb{N}$. We say that X has *counting dimension at most* (N, d, e) if there exists a definable function $f : X \to \mathcal{O}_K^d$ such that for every s,

$$X_s \xrightarrow{f} \mathcal{O}_K^d \xrightarrow{\text{proj}} \left(\frac{\mathcal{O}_K}{\mathcal{M}_K^{e(s)}} \right)^d$$

has finite fibres of size at most $N(s)$.

Counting dimension in $\mathbb{C}((t))$ and $\mathbb{Q}_p$

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An h-minimal Pila–Wilkie theorem

Let $K = k((t))$ or $K = \mathbb{Q}_p$ in a language $\mathcal{L}$ such that $\mathrm{Th}_{\mathcal{L}}(K)$ is 1-h-minimal. Assume that acl $=$ dcl and that the subgroup of b-th powers has finite index in $k^\times$, for some $b > 1$.

Theorem (Cluckers–Halupczok–Rideau-Kikuchi-V, Cantoral-Farfan–Nguyen-V)

Let $X \subset K^n$ be a curve. For every $\varepsilon > 0$ there exists a N_ε such that the counting dimension of X^{trans} is bounded by

$$(N_\varepsilon, 1, \lceil s \cdot \varepsilon \rceil).$$

Assumptions

1-h-minimality.

acl $=$ dcl.

b-th powers finite index in $k^\times$.

Two ingredients in the proof

T_r-Parametrizations.

The determinant method.

Necessity of counting dimension

Consider $\mathbb{Q}_p((t))$ with valuation ring $\mathbb{Q}_p[[t]]$. There is a 1-h-minimal structure on $\mathbb{Q}_p((t))$ in which

$$\exp : p\mathbb{Z}_p + t\mathbb{Q}_p[[t]] \to \mathbb{Q}_p((t))$$

is definable. Let $X = \mathrm{graph}(\exp)$.

Some references

Hensel minimality I, Cluckers, Halupczok and Rideau-Kikuchi, https://arxiv.org/abs/1909.13792.

Hensel minimality II; Mixed characteristic and a Diophantine application, Cluckers, Halupczok, Rideau-Kikuchi and Vermeulen, https://arxiv.org/abs/2104.09475.

A Pila–Wilkie theorem for Hensel minimal curves, Cantoral-Farfán, Huu Nguyen and Vermeulen, https://arxiv.org/abs/2107.03643.

Questions?