Valued Differential Fields

Valuations

K : a field

ordered abelian grou

valuation on K : a map $v : K \longrightarrow \Gamma \cup \{\infty\}$

such that $\quad va = \infty \iff a = 0$

$$v(a+b) \geqslant \min(va, vb)$$
$$v(ab) = va + vb$$

Example : Hahn fields $K = k((t^\Gamma))$ where k is a field. Its elements are the formal series $a = \sum_{\gamma \in \Gamma} c_\gamma t^\gamma$ with coefficients $c_\gamma \in$ exponents $\gamma \in \Gamma$, and well-ordered support

$$\operatorname{supp}(a) = \{\gamma \in \Gamma : c_\gamma \neq 0\}$$

The valuation $v : K \to \Gamma \cup \{\infty\}$ is given by

$$va = \min \operatorname{supp}(a) \qquad (a \neq 0)$$

Let K be a valued field with valuation $v : K \to \Gamma \cup \{\infty\}$. Some notation:

$\mathcal{O} := \{a \in K : va \geqslant 0\}$, the *valuation ring*

$\mathcal{o} := \{a \in K : va > 0\}$, its *maximal ideal*

$k := \mathcal{O}/\mathcal{o}$, its *residue field*.

K is a topological field with basic neighborhood of 0 given by $\{a : va > \gamma\}$ $(\gamma \in \Gamma)$

$$a \succsim b \iff b \precsim a :\iff va \leq vb$$

"a dominates b" "b is dominated by a"

$$a \succ b \iff va < vb$$

"a strictly dominates b"

So $a \precsim 1 \iff a \in \mathcal{O}$

$a \prec 1 \iff a \in \mathcal{o}$

$$a \asymp b :\iff a \precsim b \text{ and } b \precsim a \iff va = vb$$

2° Derivations

A derivation on a field K is a map $\partial : K \to K$ such that $\partial(a+b) = \partial a + \partial b$, $\partial(ab) = \partial(a)b + a\partial(b)$

Notation : $a' = \partial a$, $a'' = a^{(2)} = \partial^2 a$, $\ldots$, $a^{(n)} = \partial^n a$

$C = C_K = \{a \in K : a' = 0\}$, the constant field

$a^\dagger := \frac{a'}{a}$ for $a \neq 0$ (logarithmic derivative)

The differential field K is said to be linearly surjective if for all $a_0, \ldots, a_n, b \in K$ with $a_n \neq 0$ the equation $a_0 y + a_1 y' + \cdots + a_n y^{(n)} = b$ has a solution in K.

It follows from a theorem due to Bertrand that every differential field of characteristic 0 has a linearly surjective closure

Examples of valued differential fields

(i) Given a differential field k we make

$K = k((t^\Gamma))$ into a differential field extension b,

$$\left(\sum a_\gamma t^\gamma\right)' = \sum a_\gamma' t^\gamma$$

Here $C = C_k((t^\Gamma))$

(ii) Let k be a differential field with $\mathbb{R} \subseteq C_k$, and let $g \in K := k((t^{\mathbb{R}}))$. Then we make K into a differential field extension of k by

$$\left(\sum a_\gamma t^\gamma\right)' = \sum a_\gamma' t^\gamma + g \sum \gamma a_\gamma t^{\gamma - 1} \quad (t' = g)$$

For $g = t$ we get $\left(\sum a_\gamma t^\gamma\right)' = \sum a_\gamma' t^\gamma + \sum \gamma a_\gamma t^\gamma$

with $C = C_k$ if $(k^\times)^\dagger \cap \mathbb{R} = \{0\}$.

From now on: K is a <u>valued</u> <u>differential</u> <u>field</u>, and to avoid "positive characteristic" issues we assume that its residue field k has characteristic 0, that is $\mathcal{O} \supseteq \mathbb{Q}$. Two very different behaviours

(1) K <u>monotone</u> $\overset{\text{def}}{\Longleftrightarrow}$ $a' \preceq a$ for all a

($\therefore$ ∂ is continuous) Holds in Example (i) where $K = k((t^{\Gamma}))$ with derivation on k extended to K by $(t^{\gamma})' = 0$ for all γ. Also holds in Example (ii) for $g \preceq t$

(2) K <u>asymptotic</u> $\overset{\text{def}}{\Longleftrightarrow}$ for all $a, b \succ 1$,

$$a \preceq b \Longleftrightarrow a' \preceq b'$$

($\therefore$ ∂ is continuous) Holds in Hardy fields, the field of transseries, and in Example (ii) for $g \succ t$

Always $C \subseteq \mathcal{O}$ in the asymptotic case.

We are only interested when the derivation $\partial : K \to K$ is <u>continuous</u>, which is equivalent to: $\partial \mathfrak{o} \subseteq a\mathfrak{o}$ for some $a \in K^\times$ $(\Longleftrightarrow (a^{-1}\partial)\,\mathfrak{o} \subseteq \mathfrak{o})$

For us ∂ is as good as any multiple $a^{-1}\partial$, so we also assume from now on that ∂ has been normalized so that $\partial\mathfrak{o} \subseteq \mathfrak{o}$. (Is already the case when K is monotone.)

It follows that $\partial\mathcal{O} \subseteq \mathcal{O}$, and so ∂ induces a derivation on the residue field $\mathbf{k} = \mathcal{O}/\mathfrak{o}$:

$$a + \mathfrak{o} \mapsto a' + \mathfrak{o}.$$

It is best to choose the normalized ∂ such that this induced derivation on $\mathbf{k}$ is <u>nontrivial</u> (not always possible)

Algebraic Extensions

Let $L \supseteq K$ be an algebraic field extension, let $v : K^{\times} \to \Gamma$ be extended to a valuation $v_L : L^{\times} \to \Gamma_L \supseteq$ (always possible) and let $\partial : K \to K$ be extended to a derivation $\partial_L : L \to L$ (there is a unique such ∂_L)

Then $\partial_L \, \mathcal{O}_L \subseteq \mathcal{O}_L$, and so this makes L into a <u>bona fide</u> valued differential field extending K. Moreover:

K asymptotic $\Longrightarrow$ L asymptotic

K monotone $\Longrightarrow$ L monotone

Immediate Extensions

An *immediate* extension of a valued field E is a valued field extension with same value group, and (after natural identification) same residue field.

By Zorn, every valued field E has a maximal immediate extension F; such F is necessarily *spherically complete* (Krull): every nest of closed balls in F has nonempty intersection.

Thm Let K be a valued differential field such that the induced derivation on $\mathbb{k}$ is nontrivial. Let L be a maximal immediate valued differential field extension of K (exists by Zorn). Then L is spherically complete.

Also true for asymptotic K satisfying:

(a) K is H-asymptotic: $a \succ b \succ 1 \Rightarrow a^\dagger \succcurlyeq b^\dagger$

(b) Γ is divisible and K has asymptotic integration: for all $a \in K^\times$ there is $b \in K^\times$ such that $a \asymp b'$

Differential Polynomials

Let E be a differential ring. A differential polynom over E is a polynomial

$$P(Y) = p(Y, Y', Y'', \ldots) \text{ in the indeterminates}$$

$$Y, Y', Y'', \ldots, Y^{(n)} \text{ with coefficients in } E.$$

They make up the differential ring

$$E\{Y\} = E[Y, Y', Y'', \ldots]$$

whose derivation extends that of E and satisfies $(Y^{(n)})' := Y^{(n+1)}$.

Given such $P(Y) = p(Y, Y', Y'', \ldots)$ and $y \in E$ we set $P(y) := p(y, y', y'', \ldots)$.

Let $P = P(Y) \in K\{Y\}^{\neq 0}$,

$$P = \sum_{\vec{i}} a_{\vec{i}} Y^{\vec{i}}, \qquad \vec{i} = (i_0, \ldots, i_n)$$

$$Y^{\vec{i}} = Y^{i_0} (Y')^{i_1} \cdots (Y^{(n)})^{i_n}$$

We extend $v : K^\times \to \Gamma$ to $v : K\{Y\}^{\neq 0} \to \Gamma$

by $v(P) := \min_{\vec{i}} v(a_{\vec{i}})$

In order to study $y \mapsto P(y) : K \to K$ we use as a tool the function $\gamma \mapsto v_P(\gamma) : \Gamma \to \Gamma$ which is defined as follows:

for $g \in K^\times$, $P_{\times g} := P(gY)$

and $v(P_{\times g})$ depends only on $\gamma = vg$ (given P)

so we can define $v_P(\gamma) := v(P_{\times g})$.

If P homogeneous of degree d, then

$$v_P(\gamma) = v(P) + d\gamma + o(\gamma),$$

in particular v_P is strictly increasing when $d>0$.

(For monotone K, $v_P(\gamma) = v(P) + d\gamma$)

<u>Equalizer Theorem</u> $(P, Q \in K\{Y\}^{\neq 0})$

If P, Q homogeneous of degrees $d>e$
and Γ is divisible (enough that $(d-e)\Gamma = \Gamma$)
then there is a unique $\gamma \in \Gamma$ such that

$$v_P(\gamma) = v_Q(\gamma).$$

For $e=0$ this gives: if P is homogeneous of degree d
and $d\Gamma = \Gamma$, then $v_P : \Gamma \to \Gamma$ is bijective.

Dominant Part

Let $P \in K\{Y\}^{\neq 0}$. Dividing P by some $a \in K^\times$ with $v(a) = v(P)$ we get $v(a^{-1}P) = 0$, so $a^{-1}P$ has all its coefficients in $\mathcal{O}$, and thus has an image in $k\{Y\}^{\neq}$, called the <u>dominant part</u> of P and denoted by D_P; it is determined up to a factor from $k^\times$.

The degree $\deg D_P$ is therefore independent of the initial choice of $a \in K^\times$, and is called the <u>dominant degree</u> of P

<u>Notation</u>: $\operatorname{ddeg} P := \deg D_P$.

For $g \in K^\times$, $\operatorname{ddeg} P_{\times g}$ depends only on $\gamma = vg$, so we have a function $\operatorname{ddeg}_P : \Gamma \to \mathbb{N}$ given by $\operatorname{ddeg}_P(\gamma) = \operatorname{ddeg} P_{\times g}$. This function $\operatorname{ddeg}_P : \Gamma \to \mathbb{N}$ takes only finitely many values and is decreasing.

K is said to be <u>differential-henselian</u> if every $P \in K\{Y\}^{\neq 0}$ with $\operatorname{ddeg} P = 1$ has a zero in $\mathcal{O}$.

(For $P \in K[Y]^{\neq 0}$ this just says that K is henselian in the usual sense.)

$P = P(0) + A + R$, $A = a_0 Y + a_1 Y' + \cdots + a_n Y^{(n)}$

all terms in R have degree ≥ 2

Then

$$\operatorname{ddeg} P = 1 \iff v P(0) \geq v(A) < v(R)$$

<u>Lifting Theorem</u> If K is differential-henselian, then the differential residue field $\boldsymbol{k}$ can be lifted to a differential subfield of K contained in $\mathcal{O}$.

Connection to linear surjectivity

(i) differential-Henselian $\Longrightarrow$ linearly surjective

(ii) K differential-henselian

$\Longleftrightarrow$ (requires Equaliser Theorem)

k is linearly surjective and for every $P = P(0) + A + R$ as before, if $v(P(0)) > v(A) \leq v(R)$, then P has a zero in

(iii) Analogue of Hensel's Lemma:

$\left.\begin{array}{l}k \text{ linearly surjective} \\ K \text{ spherically complete}\end{array}\right\} \Longrightarrow$ K differential-henselia

Revisiting Scanlon's thesis

Consider K as a 3-sorted structure, with the differential residue field k and the ordered value group Γ as the two extra sorts. Now $v(C^\times)$ is a definable subgroup of Γ, which could cause trouble if it is a "wild" subgroup of Γ. Scanlon assumed K monotone, $v(C^\times) = \Gamma$: this holds in Example (i) where all t^γ are constants.

Th (Scanlon) For monotone differential-henselian K with $v(C^\times) = \Gamma$, $\mathrm{Th}(K)$ is determined by $\mathrm{Th}(k)$ and $\mathrm{Th}(\Gamma)$, in particular $K \equiv k((t^\Gamma))$ as in Example (i)

(He also assumed $K^\times$ is divisible, but this is not needed

Purely algebraic consequence:

Suppose K is monotone, differential-henselian, and $v(C^\times)=$
Then every algebraic extension of K is differential-henseli

<u>Pf</u>: extensions of finite degree of $k((t^\Gamma))$ are spherically complete, and thus differential-henselia

One can elaborate Scanlon's theorem to a <u>relative QE</u> for monotone differential-henselian K with $v(C^\times)=\Gamma$ (and an angular component map compatible with the derivation) with consequences such as:

Inside K as a 3-sorted structure,

<u>k and Γ are stably embedded and orthogonal</u>,

K has NIP $\iff$ k and Γ have NIP

Scanlon's assumption $v(C^\times) = \Gamma$ is one way to avoid trouble. What of $v(C^\times) = \{0\}$?

<u>Thm</u> Let K be monotone and $v(C^\times) = \{0\}$. Then

K is differential-henselian $\Longleftrightarrow$ K has no proper immediate differentially algebraic extension

(False if $v(C^\times) = \{0\}$ is replaced by $v(C^\times) = \Gamma$.)

Proof is based on a suggestion by Joris van der Hoeven in a different (asymptotic) setting.

This result can be used to deal with

$K = k((t^{\mathbb{R}}))$ as in Example (ii), with $t' = t$, where $k \supseteq \mathbb{R}$ has constant field $\mathbb{R}$, $(k^{\times})^{\dagger} \cap \mathbb{R} = \{0\}$ (hence $C = \mathbb{R}$, so $v(C^{\times}) = \{0\}$), and k is linearly surjecti

(These conditions are satisfied for $k =$ diff. field of purely logarithmic transseries.)

Consider this K as a 1-sorted structure with a predicate for the subfield k and a function symbol for the map $r \mapsto t^r : \mathbb{R} \to K$

On the plane to Paris I verified:

Th(K) <u>is</u> <u>axiomatized</u> <u>by</u>:

- K monotone, $v(C^{\times}) = \{0\}$
- k is a lifting of the residue field, $C = C_k$
- $c \mapsto t^c : C \to K$ is a cross-section of the valuation, with $(t^c)' = ct^c$
- K is differential-henselian
- Th(k), which includes that its constant field is real closed