Effective Chabauty and the cursed curve

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Section 1

Introduction

Rational points

$X/\mathbf{Q}$ a smooth projective curve of genus $g > 1$.

Given by (singular) plane model $f(x,y) = 0$.

Theorem (Faltings '83)

The set $X(\mathbf{Q})$ of rational points on X is finite.

Usually points are easily found by a search (if they exist).

Example ($g = 4$)

$$f(x,y) = y^3 - (x^5 - 2x^4 - 2x^3 - 2x^2 - 3x)$$
$$X(\mathbf{Q}) \supset \{(1,-2),(0,0),(-1,0),(3,0),\infty\}$$

Problem

How to prove that these are all points?

Chabauty's theorem

J will denote the Jacobian variety of X, i.e. divisors of degree 0 modulo divisors of functions. Note that J is naturally an abelian variety.

Theorem (Mordell-Weil)

$J(\mathbf{Q})$ is a finitely generated abelian group.

Given a point $b \in X(\mathbf{Q})$, we get an embedding $X(\mathbf{Q}) \to J(\mathbf{Q})$:

$$P \mapsto (P) - (b)$$

Theorem (Chabauty '41)

Let r be the rank of $J(\mathbf{Q})$. If $r < g$ then $X(\mathbf{Q})$ is finite.

Coleman: can make this effective using p-adic line integrals.

Section 2

Coleman integrals

Coleman integrals

Let:

- p a prime at which X has good reduction,
- $P, Q \in X(\mathbf{Q}_p)$,
- ω a 1-form on $X_{\mathbf{Q}_p}$ (more generally on some wide open of a rigid analytic space).

In the 80's Coleman defined path independent line integrals

$$\int_P^Q \omega$$

which can be extended to integrate over $D \in J(\mathbf{Q}_p)$, where J is the Jacobian of X (above: $D = (Q) - (P)$).

Properties

The Coleman integral has the following properties:

1. Linearity: $\int_P^Q (a\omega_1 + b\omega_2) = a\int_P^Q \omega_1 + b\int_P^Q \omega_2$.
2. Additivity in endpoints: $\int_P^Q \omega = \int_P^R \omega + \int_R^Q \omega$.
3. Change of variables: If $V' \subset X'$ is a wide open subspace of a rigid analytic space X' and $\phi : V \to V'$ a rigid analytic map then $\int_P^Q \phi^*\omega = \int_{\phi(P)}^{\phi(Q)} \omega$.
4. Fundamental theorem of calculus: $\int_P^Q df = f(Q) - f(P)$ for f a rigid analytic function on V.

A residue disk on $X_{\mathbf{Q}_p}$ is the inverse image under reduction mod p of a single point. Coleman integrals within a single residue disk are called tiny.

Tiny integrals

Let: $P, Q \in X(\mathbf{Q}_p)$ points in the same residue disk, $\omega \in H^0(X_{\mathbf{Q}_p}, \Omega^1)$.

Then $\int_P^Q \omega$ can be computed by expanding ω in a local coordinate t on the disk:

$$\omega = \sum_{i \geq 0} c_i t^i dt$$

and integrating as usual

$$\int_{t(P)}^{t(Q)} \sum_{i \geq 0} c_i t^i dt = \sum_{i \geq 0} \frac{c_i}{i+1} (t(Q)^{i+1} - t(P)^{i+1}).$$

When P and Q not in the same residue disk, does not work: series do not converge.

Analytic continuation fails over $\mathbf{Q}_p$. Coleman: use Frobenius action on p-adic cohomology.

p-adic cohomology

Can construct p-adic cohomology space $H^1_{rig}(X_{\mathbf{Q}_p})$:

- a vector space over $\mathbf{Q}_p$ isomorphic to $H^1_{\mathrm{dR}}(X_{\mathbf{Q}_p})$,
- with action F_p^* of p-th power Frobenius F_p on $X_{\mathbf{F}_p}$.

Let $U \subset X$ be an open such that $X - U$ is smooth over $\mathbf{Z}_p$ and $\omega_1, \ldots, \omega_{2g} \in \Omega^1(U_{\mathbf{Q}_p})$ a basis for $H^1_{\mathrm{dR}}(X_{\mathbf{Q}_p})$.

Then there exist:

- a matrix $\Phi \in M_{2g \times 2g}(\mathbf{Q}_p)$,
- (overconvergent) functions $f_1, \ldots, f_{2g}$ on some open of $X_{\mathbf{Q}_p}$,

such that

$$\mathrm{F}_p^*(\omega_i) = df_i + \sum_{j=1}^{2g} \Phi_{ij}\omega_j \qquad \text{for } i = 1, \ldots, 2g.$$

We can take $\omega_1, \ldots, \omega_g$ to be a basis for $H^0(X_{\mathbf{Q}_p}, \Omega^1)$.

General integrals

Recall that

$$F_p^*(\omega_i) = df_i + \sum_{j=1}^{2g} \Phi_{ij}\omega_j \qquad \text{for } i = 1, \ldots, 2g.$$

Assume that $F_p(P) = P$ and $F_p(Q) = Q$ (Teichmüller points). No loss of generality, can correct with tiny integrals. Integrating, we find

$$\int_P^Q \omega_i = \int_{F_p(P)}^{F_p(Q)} \omega_i = \int_P^Q F_p^*(\omega_i) = f_i(Q) - f_i(P) + \sum_j \Phi_{ij} \int_P^Q \omega_j.$$

So we can determine the $\int_P^Q \omega_i$ by solving the linear system

$$(\Phi - I) \int_P^Q \omega_i = f_i(P) - f_i(Q) \qquad \text{for } i = 1, \ldots, 2g.$$

Implementation

We have developed and implemented (in Magma) algorithm to compute action of F_p^* on $H_{\mathrm{rig}}(X_{\mathbf{Q}_p})$ for any X for almost all p. Application in mind: computing zeta function $Z(X_{\mathbf{F}_p}, T)$.

Package is called pcc. Can be found on our website and GitHub. Comes with Magma since v2.23, commands are: ZetaFunction and LPolynomial.

In joint work with J. Balakrishnan we have extended this to an algorithm and implementation for computing (single) Coleman integrals on arbitrary curves.

The package is called Coleman and can again be found on our website and GitHub.

Section 3

Effective Chabauty

Chabauty-Coleman

Assume a point $b \in X(\mathbf{Q})$ is known and embed $X \hookrightarrow J$ into its Jacobian by $P \mapsto (P) - (b)$.

Theorem (Chabauty-Coleman)

Let r denote the Mordell-Weil rank of J and suppose that $r < g$. Then there exists $\omega \in H^0(X_{\mathbf{Q}_p}, \Omega^1)$ such that $\int_b^P \omega = 0$ for all $P \in X(\mathbf{Q})$.

Sketch of proof.

$$
\begin{array}{ccccc}
X(\mathbf{Q}) & \longrightarrow & X(\mathbf{Q}_p) & \xrightarrow{AJ_b} & \\
\downarrow & & \downarrow & & \searrow \\
J(\mathbf{Q}) & \longrightarrow & J(\mathbf{Q}_p) & \xrightarrow{D \mapsto \int_D} & H^0(X_{\mathbf{Q}_p}, \Omega^1)^*
\end{array}
$$

$X(\mathbf{Q})$ lands in a subspace of $H^0(X_{\mathbf{Q}_p}, \Omega^1)^*$ of dimension at most r. □

Effective Chabauty

The integral $\int_b^P \omega$ can be expanded in a power series with a finite number of zeros on every residue disk.

This proves Mordell's conjecture in the case $r < g$ as already noted by Chabauty.

Since we can compute Coleman integrals, when $r < g$ this gives an algorithm to find a finite subset

$$X(\mathbf{Q}_p)_1 \subset X(\mathbf{Q}_p)$$

which contains $X(\mathbf{Q})$.

We have also implemented a basic version of this algorithm.

Explicit effective Chabauty

1. Suppose an upper bound $R < g$ is known on the rank r of J.
2. Take as input points $P_1, \ldots, P_k \in X(\mathbf{Q})$.
3. Determine the subspace S of $\omega \in H^0(X_{\mathbf{Q}_p}, \Omega^1)$ such that

$$\int_{P_1}^{P_i} \omega = 0 \qquad \text{for } i = 1, \ldots, k.$$

4. If $\dim S \leq g - R$ then for all $\omega \in S$ and $P \in X(\mathbf{Q})$

$$\int_{P_1}^{P} \omega = 0 \qquad \text{for } i = 1, \ldots, k.$$

5. Expand these conditions in power series and find the candidate points on every residue disk of $X_{\mathbf{Q}_p}$.

Example

Let us return to the example $f(x,y) = y^3 - (x^5 - 2x^4 - 2x^3 - 2x^2 - 3x)$. The Magma function *RankBounds()* proves that the rank of J is 1. This uses work of Poonen-Schaefer ('97). Now we use our code:

```
> load "coleman.m";
> Q:=y^3 - (x^5 - 2*x^4 - 2*x^3 - 2*x^2 - 3*x);
> p:=7;
> N:=15;
> data:=coleman_data(Q,p,N);
> Qpoints:=Q_points(data,1000); // PointSearch
> #vanishing_differentials(Qpoints,data:e:=50);
3
> #effective_chabauty(data,1000:e:=50),#Qpoints;
5 5
```

This proves that our list of rational points is complete.

Problems

Does not always work.

Some examples of what can go wrong:

- No upper bound on rank r. Assume some conjectures?
- $(P) - (Q)$ with $P, Q \in X(\mathbf{Q})$ do not generate full rank subgroup of $J(\mathbf{Q})$. Then $\dim S \leq g - R$ is never satisfied. Use more general $D \in J(\mathbf{Q})$? Currently, only points in $X(\mathbf{Q}_p)$ allowed.
- Too many points found: $X(\mathbf{Q}_p)_1$ strictly larger than $X(\mathbf{Q})$. Use other prime p, combine with Mordell-Weil sieving?
- Rank r known but $r \geq g$. Method as explained so far breaks down. However, recently some succes with non-abelian effective Chabauty.

What is non-abelian effective Chabauty? Let's see an example.

Section 4

The cursed curve

Serre uniformity

If $E/\mathbf{Q}$ elliptic curve and ℓ prime number then we get a residual Galois representation:

$$\rho_{E,\ell} : \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{Aut}(E[\ell]) \cong Gl_2(\mathbf{F}_\ell).$$

Theorem (Serre '72)

If E does not have complex multiplication (CM) then $\rho_{E,\ell}$ is surjective for all primes $\ell \gg 0$.

Problem (Serre)

Is the same true uniformly in E, i.e. is there a constant ℓ_0 such that $\rho_{E,\ell}$ is surjective for all elliptic curves $E/\mathbf{Q}$ without CM and all primes $\ell > \ell_0$?

When $\rho_{E,\ell}$ is not surjective, its image its contained in 1) a Borel subgroup, 2) an exceptional subgroup or 3) the normaliser of a (split or non-split) Cartan subgroup of $Gl_2(\mathbf{F}_\ell)$.

The cursed curve

Split Cartan modular curve of level 13:

$$X_s(13) = X(13)/C_s(13)^+$$

where $C_s(13)^+$ is the normaliser of a split Cartan subgroup of $GL_2(\mathbf{F}_{13})$.
Points of $X_s(13)$ are elliptic curves E with $\mathrm{Im}(\rho_{E,13}) \subset C_s(13)^+$.

Baran '14 found a defining equation, which we can rewrite as

$$y^4+5x^4-6x^2y^2+6x^3+26x^2y+10xy^2-10y^3-32x^2-40xy+24y^2+32x-16y=0.$$

The closure in $\mathbf{P}^2_{\mathbf{Q}}$ is a smooth plane quartic, so $g=3$. Jacobian is simple and by known instance of BSD and computation one finds $r=3$.

One easily finds the following seven points (in homogeneous coordinates $(X:Y:Z)$ with $x=X/Z$, $y=Y/Z$):

$$\{(1:1:1),(1:1:2),(0,0,1),(-3,3,2),(1,1,0),(0,2,1),(-1,1,0)\}.$$

Rational points

Theorem (J. Balakrishnan, N. Dogra, S. Müller, J. Tuitman, J. Vonk '17)

The rational points on $X_s(13)$ are the seven known ones (six CM points and one cusp).

Preprint: `https://arxiv.org/abs/1711.05846`.

Corollary

There does not exist an elliptic curve $E/\mathbf{Q}$ without CM such that the image of its mod ℓ Galois representation is contained in the normalizer of a split Cartan subgroup of $GL_2(\mathbf{F}_\ell)$ for $\ell = 13$.

For all $\ell \neq 13$ it was already known (Bilu-Parent-Rebolledo '11) whether such elliptic curves exist or not (for $\ell \leq 7$ yes, otherwise no). For Borel and exceptional cases answer known by Mazur '78 and Serre '72. Non-split Cartan case is (mostly) open!

Section 5

Non-abelian Chabauty on $X_s(13)$

Non-abelian Chabauty

For $X = X_s(13)$, the map $D \to \int_D$ gives an isomorphism

$$J(\mathbf{Q}) \otimes \mathbf{Q}_p \to H^1_{\mathrm{dR}}(X_{\mathbf{Q}_p})^*.$$

Therefore, we cannot find the global points among the local ones using linear relations in the Abel-Jacobi map.

The idea of Kim's non-abelian Chabauty program is to refine the Abel-Jacobi map, by replacing linear relations by higher degree ones.

$$X(\mathbf{Q}_p) \supset X(\mathbf{Q}_p)_1 \supset X(\mathbf{Q}_p)_2 \supset \ldots \supset X(\mathbf{Q})$$

For $X_s(13)$ it turns out that $X(\mathbf{Q}_p)_2 = X(\mathbf{Q})$ for $p = 17$.

We always assume that X has potentially good reduction everywhere, which is the case for $X_s(13)$.

The general idea (I)

The Chabauty diagram we saw before can be extended as follows:

$$
\begin{array}{ccccc}
X(\mathbf{Q}) & \longrightarrow & X(\mathbf{Q}_p) & & \\
\downarrow & & \downarrow & \overset{\mathrm{AJ}_b}{\searrow} & \\
J(\mathbf{Q}) & \longrightarrow & J(\mathbf{Q}_p) & \xrightarrow{D \mapsto \int_D} & H^0(X_{\mathbf{Q}_p}, \Omega^1)^* \\
\downarrow \kappa & & \downarrow \kappa_p & & \downarrow \simeq \\
H^1_{\mathrm{f}}(G_T, V) & \xrightarrow{\mathrm{loc}_p} & H^1_{\mathrm{f}}(G_p, V) & \xrightarrow{\simeq} & V_{\mathrm{dR}}/\mathrm{Fil}^0
\end{array}
$$

where:

- T_0 the set of primes of bad reduction and $T = T_0 \cup \{p\}$,
- $G_p = \mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ and G_T the maximal quotient of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ unramified outside T,
- $V = H^1_{\text{ét}}(\overline{X}, \mathbf{Q}_p)^*$, $V_{dR} = H^1_{\mathrm{dR}}(X_{\mathbf{Q}_p})^*$, Fil = (dual) Hodge filtration,
- H^1_{f} and κ, κ_p: Bloch Kato Selmer groups and Kummer maps.

The general idea (II)

Note that $V = H^1_{\text{ét}}(\overline{X}, \mathbf{Q}_p)^*$ is the maximal abelian quotient of the $\mathbf{Q}_p$ étale fundamental group $\pi_1^{dR}(\overline{X}, b)$.

M. Kim proposes to cut out the middle row in the diagram and replace V by the maximal n-unipotent quotients U_n of $\pi_1^{dR}(\overline{X}, b)$:

$$\begin{array}{ccccc} X(\mathbf{Q}) & \longrightarrow & X(\mathbf{Q}_p) & & \\ \downarrow j_n^{\text{ét}} & & \downarrow j_{n,p}^{\text{ét}} & \searrow j_n^{\text{dR}} & \\ H^1_{\mathrm{f}}(G_T, \mathrm{U}_n^{\text{ét}}) & \xrightarrow{\mathrm{loc}_{n,p}} & H^1_{\mathrm{f}}(G_p, \mathrm{U}_n^{\text{ét}}) & \xrightarrow{\simeq} & \mathrm{U}_n^{\mathrm{dR}}/\mathrm{Fil}^0. \end{array}$$

He defines $X(\mathbf{Q}_p)_n = (j_{n,p}^{\text{ét}})^{-1}(\mathrm{loc}_{n,p}(H^1_{\mathrm{f}}(G_T, \mathrm{U}_n^{\text{ét}})))$, so that:

$$X(\mathbf{Q}_p) \supset X(\mathbf{Q}_p)_1 \supset X(\mathbf{Q}_p)_2 \supset \ldots \supset X(\mathbf{Q})$$

$H^1_{\mathrm{f}}(G_T, \mathrm{U}_n^{\text{ét}})$ and $H^1_{\mathrm{f}}(G_p, \mathrm{U}_n^{\text{ét}})$ are naturally schemes. If $\mathrm{loc}_{n,p}$ is not dominant then $X(\mathbf{Q}_p)_n$ is finite (analog to $r < g$ for $n = 1$)! We will only use $n = 2$ and take a further quotient $\mathrm{U}_Z^{\text{ét}}$ of $\mathrm{U}_2^{\text{ét}}$.

Quadratic Chabauty functions

Fix $b \in X(\mathbf{Q})$. We want to find quadratic Chabauty functions $\theta : X(\mathbf{Q}_p) \to \mathbf{Q}_p$ such that:

1. On each residue disk, the map

$$(AJ_b, \theta) : X(\mathbf{Q}_p) \to H^0(X_{\mathbf{Q}_p}, \Omega^1)^* \times \mathbf{Q}_p$$

   has Zariski dense image and is given by power series.
2. There exist
   - an endomorphism E of $H^0(X_{\mathbf{Q}_p}, \Omega^1)^*$,
   - a functional $c \in H^0(X_{\mathbf{Q}_p}, \Omega^1)^*$,
   - a bilinear form $B : H^0(X_{\mathbf{Q}_p}, \Omega^1)^* \otimes H^0(X_{\mathbf{Q}_p}, \Omega^1)^* \to \mathbf{Q}_p$,

   such that, for all $x \in X(\mathbf{Q})$:

$$\theta(x) = B\left(AJ_b(x), E(AJ_b(x)) + c\right).$$

Nice correspondences

We will construct θ from a nice correspondence.

Let Z be a correspondence on X, i.e. a divisor on $X \times X$.

We denote:

- τ the involution $(x_1, x_2) \mapsto (x_2, x_1)$ on $X \times X$,
- $\pi_1, \pi_2 : X \times X \to X$ the canonical projections.

Z is symmetric if there exist $Z_1, Z_2 \in Pic(X)$ such that

$$\tau_* Z = Z + \pi_1^*(Z_1) + \pi_2^*(Z_2).$$

Induces endomorphism ξ_Z of $H^1_{\mathrm{dR}}(X)$ and class in $H^1_{\mathrm{dR}}(X_{\mathbf{Q}_p}) \otimes H^1_{\mathrm{dR}}(X_{\mathbf{Q}_p})$.

Z is nice if nontrivial, symmetric and $\mathrm{Tr}(\xi_Z) = 0$.

Unipotent overconvergent F-isocrystal

Let $Y = X - x^{-1}(\infty)$. Take $\vec{\omega} = \{\omega_1, \ldots \omega_6\}$ a basis of $V_{dR} = H^1_{\mathrm{dR}}(X)$.

Put the connection $\nabla = d + \Lambda$ on $\mathcal{A}_Z(b) = \mathcal{O}_Y \oplus \mathcal{O}_Y^{\oplus 6} \oplus \mathcal{O}_Y$:

$$\Lambda = - \begin{pmatrix} 0 & 0 & 0 \\ \vec{\omega} & 0 & 0 \\ \eta & \vec{\omega}^t Z & 0 \end{pmatrix}$$

where η has to satisfy: 1) it is logarithmic 2) ∇ extends to a holomorphic connection on X.

By a crystalline comparison theorem $\mathcal{A}_Z(b)$ admits a Frobenius structure, i.e. an isomorphism

$$F : \mathrm{F}_p^* \mathcal{A}_Z(b) \to \mathcal{A}_Z(b)$$

horizontal w.r.t ∇, turning $(\mathcal{A}_Z(b), \nabla)$ into a unipotent overconvergent F-isocrystal.

Frobenius structure

Let $\tilde{b}$ be the Teichmüller lift of b, i.e. $\tilde{b} \equiv b \bmod p$ and $\mathrm{F}_p(\tilde{b}) = \tilde{b}$. The inverse of the matrix of the Frobenius structure F is given by

$$G = \begin{pmatrix} 1 & 0 & 0 \\ \vec{f} & \Phi & 0 \\ h & \vec{g}^t & p \end{pmatrix} \qquad \text{such that} \qquad dG = G\Lambda - \mathrm{F}_p^*(\Lambda)G.$$

The differential equation is equivalent to:

$$\begin{aligned} \mathrm{F}_p^*\vec{\omega} &= d\vec{f} + \Phi\vec{\omega} & f(\tilde{b}) &= 0 \\ d\vec{g}^t &= d\vec{f}^t Z\Phi & & \\ dh &= \vec{\omega}^t\Phi^t Z\vec{f} + d\vec{f}^t Z\vec{f} - \vec{g}^t\vec{\omega} + \mathrm{F}_p^*\eta - p\eta & h(\tilde{b}) &= 0 \end{aligned}$$

and can be solved using (an adaptation of) our algorithms for computing Coleman integrals!

Construct θ_Z

For any $x \in X(\mathbf{Q}_p)$ can pull back $A_Z(b)$:

$$A_Z(b,x) = x^*(A_Z(b)).$$

This is a mixed extension of filtered ϕ-modules in the sense of p-adic Hodge theory. Note that the action of ϕ is given by

$$T_{\tilde{x},x} \circ G^{-1}(\tilde{x}) \circ T_{b,\tilde{b}}$$

where $T_{x,y}$ denotes parallel transport from x to y.

For such objects, Nekovar defines a p-adic height function $h_p()$. We set

$$\theta_Z(x) = h_p(A_Z(b,x)).$$

For any nice correspondence Z we have that θ_Z is a quadratic Chabauty function (with $E = \xi_Z$, c is explicit as well).

Computing h_p (I)

Let $s_1, s_2 : V_{dR} \to V_{dR}$ be projections splitting the (dual) Hodge filtration on V_{dR} where s_1 corresponds to V_{dR}/Fil^0 and s_2 to Fil^0.

Moreover, let

$$s_0 : \mathbf{Q}_p \oplus V_{dR} \oplus \mathbf{Q}_p(1) \xrightarrow{\sim} A_Z(b,x).$$

be a splitting of vector spaces and choose further splittings:

$$\begin{aligned} s^{\phi} &: \mathbf{Q}_p \oplus V_{dR} \oplus \mathbf{Q}_p(1) &\xrightarrow{\sim} &\quad A_Z(b,x), \\ s^{Fil} &: \mathbf{Q}_p \oplus V_{dR} \oplus \mathbf{Q}_p(1) &\xrightarrow{\sim} &\quad A_Z(b,x), \end{aligned}$$

where s^{ϕ} is Frobenius equivariant and s^{Fil} respects the filtrations.

Computing h_p (II)

Suppose that

$$s_0^{-1} \circ s^{\phi} = \begin{pmatrix} 1 & 0 & 0 \\ \boldsymbol{\alpha}_{\phi} & 1 & 0 \\ \gamma_{\phi} & \boldsymbol{\beta}_{\phi}^{\intercal} & 1 \end{pmatrix} \qquad s_0^{-1} \circ s^{Fil} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \gamma_{Fil} & \boldsymbol{\beta}_{Fil}^{\intercal} & 1 \end{pmatrix}.$$

Then we have the following very concrete description of $\theta_Z(x)$:

$$\theta_Z(x) = h_p(A_Z(b,x)) = \gamma_{\phi} - \gamma_{Fil} - \boldsymbol{\beta}_{\phi}^{\intercal} \cdot s_1(\boldsymbol{\alpha}_{\phi}) - \boldsymbol{\beta}_{Fil}^{\intercal} \cdot s_2(\boldsymbol{\alpha}_{\phi}).$$

So computing $\theta_Z(x)$ is reduced to (a lot of rather messy) linear algebra.

On every residue disk $\theta_Z(x)$ is given by a power series.

The complete computation

Take $p = 17$.

1) For each Z:

- Determine endomorphism E and functional c.
- Compute a 1-form η such that Λ extends to all of X.
- On each residue disk compute power series $\theta(x)$, by determining Hodge data β_{Fil}, γ_{Fil} and Frobenius data $\alpha_\phi, \beta_\phi, \gamma_\phi$.
- Use 4 known points on the curve to fit the bilinear form B_Z such that:

$$\theta_Z(x) = B_Z\left(AJ_b(x), E(AJ_b(x)) + c\right) \quad (*)$$

for all $x \in X(\mathbf{Q})$.

2) On each residue disk find the common solutions to $(*)$ for $Z = Z_1, Z_2$. Check we do not get any new (candidate) points.

Some computational details

$$f(x,y) = y^4 + 5x^4 - 6x^2y^2 + 6x^3 + 26x^2y + 10xy^2 - 10y^3 - 32x^2 - 40xy + 24y^2 + 32x - 16y = 0.$$

$$\vec{\omega} := \begin{pmatrix} 1 \\ x \\ y \\ -160x^4/3 + 736x^3/3 - 16x^2y/3 + 436x^2/3 - 440xy/3 + 68y^2/3 \\ -80x^3/3 + 44x^2 - 40xy/3 + 68y^2/3 - 32 \\ -16x^2y + 28x^2 + 72xy - 4y^2 - 160x/3 + 272/3 \end{pmatrix} \frac{dx}{\left(\frac{\partial f}{\partial y}\right)}$$

We use the correspondences Z with $\xi_Z = 6a_q - \mathrm{Tr}(a_q)Id$ for $q = 7, 11$:

$$Z_1 = \begin{pmatrix} 0 & 112 & -656 & -6 & 6 & 6 \\ -112 & 0 & -2576 & 15 & 9 & 27 \\ 656 & 2576 & 0 & 3 & 3 & -3 \\ 6 & -15 & -3 & 0 & 0 & 0 \\ -6 & -9 & -3 & 0 & 0 & 0 \\ -6 & -27 & 3 & 0 & 0 & 0 \end{pmatrix} Z_2 = \begin{pmatrix} 0 & -976 & -1104 & 10 & -6 & 18 \\ 976 & 0 & -816 & -3 & 1 & 3 \\ 1104 & 816 & 0 & -3 & 3 & -11 \\ -10 & 3 & 3 & 0 & 0 & 0 \\ 6 & -1 & -3 & 0 & 0 & 0 \\ -18 & -3 & 11 & 0 & 0 & 0 \end{pmatrix}$$

Plans for the future

In increasing order of difficulty:

- Combine our implementation of (classical) Chabauty with some Mordell Weil sieving and/or descent.
- Develop and implement algorithms to compute double (perhaps triple) Coleman integrals on general curves.
- Write code to do the quadratic Chabauty computation for $X_s(13)$ in an automated way so we can do other examples (which is not the case right now).
- Work out other examples, perhaps some $X_{ns}(\ell)$?
- What about higher Chabauty, working with (quotients of) U_3 for example. How explicit can this be made?