Quasianalytic Ilyashenko Algebras

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Géométrie et Théorie des Modèles
Paris, March 10, 2017

Dulac's problem

Let $\xi : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be a real analytic vector field.

Claim

The vector field ξ has finitely many limit cycles.

Dulac's proof (1923)

(1) If there are infinitely many limit cycles, they must approach a **polycyle** $\mathcal{P}$.

(2) The **first-return map** $\varphi : (0, \epsilon) \longrightarrow (0, \epsilon)$ of $\mathcal{P}$ satisfies

$$\varphi(x) - \sum_{k=0}^{n} p_k(\log x) x^{\nu_k} = o(x^{\nu_n}) \quad \text{as } x \to 0^+$$

for $n \in \mathbb{N}$, where $0 < \nu_0 < \nu_1 < \cdots \to \infty$, each p_k is a polynomial and p_0 is a nonzero constant.

Dulac's problem

We write

$$\widehat{\varphi} := \sum_{n=0}^{\infty} p_n(\log x) x^{\nu_n}$$

for the asymptotic expansion of φ.

Dulac's proof (1923)

(3) If $\mathcal{P}$ is approached by cycles, then $\widehat{\varphi} = x$.

(4) But $\widehat{\varphi} = x$ implies $\varphi = x$, i.e., none of the cycles near $\mathcal{P}$ are limit cycles, a contradiction.

In 1981, Ilyashenko realized that $\widehat{\varphi} = x$ does not imply $\varphi = x$ in general; an additional condition is needed for this implication.

A **standard power domain** is a complex domain

$$U_C^\epsilon := \left\{z + C(1+z)^\epsilon : \ \mathrm{re}\, z > 0\right\},$$

where $C > 0$ and $0 < \epsilon < 1$.

Uniqueness Principle (Ilyashenko)

Let U be a standard power domain and $\mathbf{h} : U \longrightarrow \mathbb{C}$ be bounded and holomorphic. If

$$\mathbf{h}(x) = o\left(e^{-nx}\right) \quad \text{as } x \to +\infty, \quad \text{for all } n \in \mathbb{N},$$

then $\mathbf{h} = 0$.

Therefore, Ilyashenko considers the first-return map φ in the "logarithmic chart", that is, we look at $\varphi \circ e^{-x}$ near $+\infty$, which has asymptotic expansion $\widehat{\varphi} \circ e^{-x}$ there.

almost regular

A function $f : (a, +\infty) \longrightarrow \mathbb{R}$ is **almost regular** if there are a standard quadratic domain $U = U_C^{1/2}$, reals $0 < \nu_0 < \nu_1 < \cdots \to \infty$ and polyomials p_k such that

1. f has a bounded holomorphic extension $\mathbf{f} : U \longrightarrow \mathbb{C}$;
2. p_0 is a nonzero constant;
3. for $n \in \mathbb{N}$, we have

$$\mathbf{f}(z) - \sum_{k=0}^{n} p_k(z) e^{-\nu_k z} = o\left(e^{-\nu_n z}\right) \quad \text{as } |z| \to \infty \text{ in } U.$$

The asymptotic expansion $\widehat{f} := \sum_{n=0}^{\infty} p_n(x) e^{-\nu_n x}$ is called a **Dulac series.**

Quasianalyticity

By the Uniqueness Principle, the map T is injective.

Dulac's problem (hyperbolic case)

Examples of almost regular functions

1. e^{-x} is almost regular with $T(e^{-x}) = e^{-x}$.
2. (Ilyashenko) If $\mathcal{P}$ is hyperbolic, then $\varphi \circ e^{-x}$ is almost regular.

Corollary

Dulac's problem for hyperbolic polycycles.

Ilyashenko's proof of the general case follows the same train of thought, but is much more involved.

Remarks

1. $\mathcal{A}$ is closed under log-composition, that is, if $f, g \in \mathcal{A}$ and $1/g(x) \to +\infty$ as $x \to +\infty$, then $f \circ (-\log) \circ g$ belongs again to $\mathcal{A}$.
2. $\mathcal{A}$ is NOT closed under addition.

our motivation

Mourtada's Conjecture

If $\xi = \xi_0$ belongs to a real analytic family of vector fields ξ_ν, and if each ξ_ν has only hyperbolic singularities, then there is N $\in \mathbb{N}$ such that ξ_ν has at most N limit cycles, for all small enough ν.

Theorem (Kaiser-Rolin-S 2006)

Mourtada's conjecture holds with "non-resonant hyperbolic" in place of "hyperbolic".

We propose the following approach to Mourtada's conjecture:

1. Extend $\mathcal{A}$ into an algebra ("Ilyashenko algebra").
2. Define corresponding algebras in any number of variables.
3. Prove these algebras generate an o-minimal structure.
4. Prove that the parametric first-return maps of hyperbolic families are definable.

The remainder of this talk is about Step 1.

Step 1: the series

A **logarithmic generalized power series** is a series of the form

$$F = G\left(\frac{1}{\exp}, \frac{1}{x}, \frac{1}{\log}, \dots, \frac{1}{\log_{k-1}}\right),$$

where $\log_i$ denotes the i-th compositional iterate of log and $G(X_0, \dots, X_k)$ is a generalized power series with natural support (i.e., $\text{supp}(G) \cap B$ is finite for any bounded $B \subseteq \mathbb{R}^{k+1}$).

Example

Dulac series are logarithmic generalized power series.

the monomials

$L := \langle \exp, x, \log, \log_2, \dots \rangle^{\times}$
for $i \in \mathbb{N}$, $L_i := \langle \exp, x, \dots, \log_{i-1} \rangle^{\times}$

Note: L and L_i are multiplicative $\mathbb{R}$-vector subspaces of the Hardy field $\mathcal{H}$ of $\mathbb{R}_{\text{an,exp}}$.

Step 1: natural support

Every logarithmic generalized power series F belongs to the generalized series field $\mathbb{R}((L))$, i.e., $F = \sum_{m \in L} a_m m$ with $a_m \in \mathbb{R}$ and $\mathrm{supp}(F)$ anti-well ordered.

Actually: if F is a log. gen. power series in $\mathbb{R}((L_i))$, then $\mathrm{supp}(F)$ has order type at most ω^{i+1}.

Example

$F = \sum_{m,n \in \mathbb{N}} x^{-m} \exp^{-n}$ has support of order type ω^2.

Definition

$S \subseteq L$ is **natural** if $S \cap (a, +\infty)_L$ is finite for every $a \in L$.

Example

Dulac series belong to $\mathbb{R}((L_1))$ and have natural support.

Step 1: strong asymptotic expansions

Lemma

L is a ***scale on standard power domains****, that is, every bounded $m \in L$ has a bounded holomorphic extension* $\mathbf{m}$ *on every standard power domain.*

Not true on half-planes $H(a) := \{z \in \mathbb{C} : \ \mathrm{re}\, z > a\}$: the monomial $x \exp^{-1}$ is bounded, but its holomorphic extension on $H(a)$ is not.

Definition

$f : (a, +\infty) \longrightarrow \mathbb{R}$ has **strong asymptotic expansion** $F = \sum a_m m \in \mathbb{R}\,(\!(L_0)\!)$ if $\mathrm{supp}(F)$ is natural and there is a standard power domain U such that

1. f has a holomorphic extension $\mathbf{f} : U \longrightarrow \mathbb{C}$;
2. for $n \in L_0$ we have $\mathbf{f} - \sum_{m \geq n} a_m \mathbf{m} = o(\mathbf{n})$ in U.

Step 1: stage 0 of the construction

Let $\mathcal{A}_0$ be the set of all *bounded* germs f at $+\infty$ with a strong asymptotic expansion $T_0 f \in \mathbb{R}(\!(L_0)\!)$.

Proposition

$\mathcal{A}_0$ is a ring and $T_0 : \mathcal{A}_0 \longrightarrow \mathbb{R}(\!(L_0)\!)$ is a ring homomorphism. So by the Uniqueness Principle, T_0 is injective.

Let $\mathcal{F}_0$ be the fraction field of $\mathcal{A}_0$ with corresonding extension of T_0; then T_0 still represents strong asymptotic expansion.

Next, we right-shift $\mathcal{F}_0$ by log: set $L_1' := L_0 \circ \log$, $\mathcal{F}_1' := \mathcal{F}_0 \circ \log$ and $T_1' := T_0 \circ \log$.

Corollary

$\mathcal{F}_1'$ is a field and $T_1' : \mathcal{F}_1' \longrightarrow \mathbb{R}\left(\left(L_1'\right)\right)$ is a field homomorphism. Every $f \in \mathcal{F}_1'$ is polynomially bounded.

To iterate, we use $\mathcal{F}_1'$ as coefficients in asymptotic expansions.

Step 1: "even stronger" asymptotic expansions

$\mathcal{C}$:= set of all germs at $+\infty$ of continuous real functions
$\mathcal{C}_{\text{pol}}$:= set of all polynomially bounded germs in $\mathcal{C}$
$\mathcal{C}_{\text{pol}}^{\text{spd}}$:= set of germs in $\mathcal{C}_{\text{pol}}$ that have a polynomially bounded holomorphic extension on every standard power domain

Example

Since log maps the right half-plane $H(1)$ into the half-strip $\{z \in \mathbb{C} : \ \text{re}\, z > 0, \ |\text{im}\, z| < \pi/2\}$, every germ in $\mathcal{F}_1'$ belongs to $\mathcal{C}_{\text{pol}}^{\text{spd}}$.

Definition

$f : (a, +\infty) \longrightarrow \mathbb{R}$ has **strong asymptotic expansion** $F = \sum a_m m \in \mathcal{C}_{\text{pol}}^{\text{spd}}((L_0))$ if supp(F) is natural and there is a standard power domain U such that

1. f has a holomorphic extension $\mathbf{f} : U \longrightarrow \mathbb{C}$;
2. for $n \in L_0$ we have $\mathbf{f} - \sum_{m \geq n} \mathbf{a}_m \mathbf{m} = o(\mathbf{n})$ in U.

Step 1: stage 1 of the construction

Let $\mathcal{A}_1$ be the set of all *bounded* germs f at $+\infty$ with a strong asymptotic expansion $\tau_1 f \in \mathcal{F}'_1 ((L_0))$.

Proposition

$\mathcal{A}_1$ is a ring and $\tau_1 : \mathcal{A}_1 \longrightarrow \mathcal{F}'_1 ((L_0))$ is an injective ring homomorphism.

Let $\mathcal{F}_1$ be the fraction field of $\mathcal{A}_1$ with corresonding extension of τ_1; then τ_1 still represents strong asymptotic expansion.

Next, for $f \in \mathcal{F}_1$ with $\tau_1(f) = \sum a_m m$, set $T_1(f) := \sum T'_1(a_m)m$. Again, we right-shift $\mathcal{F}_0$ by log: set $L'_2 := L_1 \circ \log$, $\mathcal{F}'_2 := \mathcal{F}_1 \circ \log$ and $T'_2 := T_1 \circ \log$.

Corollary

$\mathcal{F}'_2$ is a field and $T'_2 : \mathcal{F}'_2 \longrightarrow \mathbb{R}((L'_2))$ is a field homomorphism. Every $f \in \mathcal{F}'_2$ is polynomially bounded.

Step 1: stage ω of the construction

Iterating this construction, we obtain fields $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots$ with injective field homomorphisms $T_i : \mathcal{F}_i \longrightarrow \mathbb{R}(\!(L_i)\!)$ such that T_{i+1} extends T_i for each i.

So we set $\mathcal{F} := \bigcup_{i \in \mathbb{N}} \mathcal{F}_i$ and let $T : \mathcal{F} \longrightarrow \mathbb{R}(\!(L)\!)$ be the common extension of all T_i.

The construction shows that we have the following: a set $S \subseteq \mathbb{R}(\!(L)\!)$ is **truncation closed** if $F_m \in S$ for every $F \in S$ and $m \in L$.

Theorem 1

1. *The image $T(\mathcal{F})$ is truncation closed and, for $f \in \mathcal{F}$ and $n \in L$, we have*

$$f - T^{-1}(T(f)_n) = o(n) \quad \text{at } +\infty.$$

2. *$\mathcal{F}$ is closed under differentiation and* log*-composition.*

So we call $(\mathcal{F}, L, T)$ a **quasianalytic asymptotic (qaa)** field.

Step 2: things get complicated

The straightforward idea for Step 2 is now to generalize this construction to several variables, using series whose monomials are products of powers of iterates of log in the various variables.
However, for Step 3 (proving o-minimality), we hope for stability of the constructed algebras under maps like blow-up substitutions.
So we need a collection of monomials that is stable under taking blow-down images of monomials.

Idea

Work with monomials definable in $\mathbb{R}_{\text{an,exp}}$.

So we need to revisit Step 1, replacing L by larger suitable subspaces of $\mathcal{H}$.

Step 1 revisited: a larger set of monomials

Let $\mathcal{B}$ be the additive $\mathbb{R}$-vector subspace of $\mathcal{H}$ consisting of all bounded germs of $\mathcal{H}$.

Remark

Let $\mathcal{U}$ be an orthogonal complement of $\mathcal{B}$ in $\mathcal{H}$. Then set $e^{\mathcal{U}}$ is a multiplicative $\mathbb{R}$-vector subspace of $\mathcal{H}^\times$ that contains exactly one representative of each Archimedean class of $\mathcal{H}$.

We are interested in one particular orthogonal complement of $\mathcal{B}$: the set $\mathcal{PI}$ of all “purely infinite” elements of $\mathcal{H}$, as defined by van den Dries, Macintyre and Marker.

Set $\mathcal{L} := e^{\mathcal{PI}}$; note that $L \subseteq \mathcal{L}$.

Theorem 2

1. *$\mathcal{L}$ is a scale on standard power domains.*
2. *There is a qaa field $(\mathcal{K}, \mathcal{L}, S)$ that extends $(\mathcal{F}, L, T)$.*