O-minimality and Hilbert’s 16th problem

Patrick Speissegger

McMaster University

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the co-authors

This is joint work with Tobias Kaiser and Jean-Philippe Rolin.

Dulac's Problem

Let $\xi : S^2 \longrightarrow TS^2$ be a real analytic vector field.

Definition

A cycle C of ξ is a **limit cycle** if C is contained in the (topological) closure of some non-compact trajectory of ξ.

Dulac's Problem

ξ has finitely many limit cycles.

Ecalle and Il'yashenko independently found proofs of Dulac's problem in the early 1990s, completing Dulac's original strategy.

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Dulac's strategy

Assume ξ has infinitely many limit cycles. Then they must pile up towards a nonempty, compact subset Γ of S^2.

Definition

Γ is called a **limit periodic set** of ξ.

Dulac showed that Γ must be a **polycycle**, that is, a closed curve composed of finitely many singular points of ξ connected by trajectories.

It remains to show that the **Poincaré first-return map** near Γ has finitely many isolated fixed points.

Hilbert's 16th Problem (second part)

If ξ is polynomial, then the number of its limit cycles has a finite upper bound that depends only on the degree of ξ.

To approach H16, one has to study not just the individual ξ, but the whole family ξ_ν of polynomial vector fields of a given degree. This leads to complications:

1. More complicated limit periodic sets.
2. Study families of Poincaré first-return maps.

H16 has a vaguely model-theoretic flavor, but no model-theoretic framework has been found to exploit this.

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finite cyclicity

Definition

$\Gamma \subseteq S^2$ is a **limit periodic set** of the *family* ξ_ν if there are limit cycles C_i of ξ_{ν_i} such that $\nu_i \to \nu$ and (C_i) converges in the sense of Hausdorff to Γ.

Roussarie's conjecture

Let $\Gamma \subseteq S^2$ be a limit periodic set of ξ_ν. Then there are $n \in \mathbb{N}$ and open neighbourhoods U of Γ in S^2 and V of ν in the parameter space such that ξ_μ has at most n limit cycles contained in U, for all $\mu \in V$.

A topological compactness argument shows that Roussarie's conjecture implies H16.

Our hope

...is to prove Roussarie's conjecture when Γ is a hyperbolic polycycle.

Definition

- A singularity p of ξ is **hyperbolic** if the linear part of ξ at p has two nonzero real eigenvalues of opposite signs.
- A polycycle of ξ is **hyperbolic** if each of its singularities is hyperbolic.

Naive approach: let Γ be a hyperbolic polycycle of ξ_0 and P_ν be the family of Poincaré first-return maps of ξ_ν near Γ, for ν near 0.

"Conjecture"

The expansion $\mathbb{R}_{P_\nu}$ of the real field by P_ν is o-minimal.

We assume 0 is a **hyperbolic** singularity of ξ.

Let γ^- and γ^+ be two adjacent separating trajectories with limit point 0; we assume γ^- is incoming and γ^+ is outgoing.

We fix two segments Λ^- and Λ^+ transverse to ξ and equipped with analytic charts *x* and *y*.

For some sufficiently small $\epsilon > 0$, we denote by $g : (0, \epsilon) \longrightarrow (0, \epsilon)$ the **transition map** of ξ from Λ^- to Λ^+.

Strategy

Prove that there exists an o-minimal expansion $\mathcal{R}$ of $\mathbb{R}_{\text{an}}$ in which all such transition maps are definable.

Theorem (Dulac and Ilyashenko)

Let g be a transition map of ξ near 0. Then there is a series $\widehat{g} = p_0 X^{\nu_0} + \sum_{j=1}^{\infty} p_j(\log X) X^{\nu_j}$ such that

(∗) *$g(e^z)$ extends analytically to a **quadratic domain***

$$W = \left\{ z \in \mathbb{C} : \operatorname{Re} z < r - C\sqrt{|\operatorname{Im} z|} \right\} \textit{ with } r \in \mathbb{R}, C > 0,$$

such that for every $n \geq 1$,

$$g(e^z) - p_0 e^{\nu_0 z} - \sum_{j=1}^{n} p_j(z) e^{\nu_j z} = o\left(e^{\nu_n \operatorname{Re} z}\right)$$

as $\operatorname{Re} z \to -\infty$ in W.

Ilyashenko functions

We call $g \in \mathcal{D}_{\log}$ satisfying $(*)$ an **Ilyashenko function**, and we let $\mathcal{I}_{\log}$ be the set of all such germs.

Theorem (Ilyashenko)

1. *If $g \in \mathcal{I}_{\log}$ is such that $\widehat{g} = X$, then $g = x$ (**quasi-analyticity**).*
2. *$\mathcal{I}_{\log}$ is closed under composition.*

Corollary

Let Γ be a polycycle of ξ such that every vertex of Γ is a hyperbolic singularity of ξ. Then ξ has finitely many limit cycles near Γ.

First results

We let $\mathcal{Q}$ be the subset of $\mathcal{I}_{\text{log}}$ consisting of all germs that do not contain log terms in their asymptotic expansions.

Theorem (1)

There is a model-complete and o-minimal expansion $\mathbb{R}_{\mathcal{Q}}$ of $\mathbb{R}_{an}$ in which every germ in $\mathcal{Q}$ are definable. In particular, every transition map near a non-resonant hyperbolic singularity of ξ is definable in $\mathbb{R}_{\mathcal{Q}}$.

What does this do for parametric families ξ_ν?

Proposition (2)

Assume ξ_ν is an analytic unfolding of ξ_0 such that each ξ_ν has only non-resonant hyperbolic singularities. Then the family of transition maps near each singularity of ξ_ν is definable in $\mathbb{R}_{\mathcal{Q}}$.

Next goals

We are now trying to extend

- Theorem (1) to all transition maps near *elementary* singular points of a single vector field ξ;
- Proposition (2) to an analytic family ξ_ν of vector fields with only hyperbolic singularities.

For both these extensions, the main difficulty lies in

- defining corresponding Ilyashenko functions in several variables;
- understanding blowings-up for these functions, in order to obtain a normalization algorithm.

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