The dynamical Mordell-Lang problem

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[1] This a report on joint work with Pietro Corvaja, Dragos Ghioca and Umberto Zannier available at arXiv:1802.05309.

Dynamical return sets

Given a semigroup S, an action $S \curvearrowright X$ on a set X, a point $a \in X$ and a target $Y \subseteq X$, the return set is $E = E(X, S, a, Y) := \{s \in S \ : \ s \cdot a \in Y\}$.

We will work mostly with the case of $S = \mathbb{N}$ so that the dynamical system $\mathbb{N} \curvearrowright X$ is determined by a single self-map $f : X \to X$ so that $n \cdot x = f^{\circ n}(x)$.

In general, there are no real restrictions on what sets may appear as return sets, but when we restrict to the category of algebraic varieties, we expect the possible return sets to be very limited.

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Dynamical Mordell-Lang conjecture in characteristic zero

Conjecture (Ghioca, Tucker)

Let K be a field of characteristic zero, X be an algebraic variety over K, $f : X \to X$ a regular self-map, $a \in X(K)$ a K-rational point on X and $Y \subseteq X$ a closed subvariety. Then the return set $E = \{n \in \mathbb{N} \ : \ f^{\circ n}(a) \in Y(K)\}$ is a finite union of arithmetic progressions $\{b + mc \ : \ m \in \mathbb{N}\}$ for appropriate natural numbers b and c (possibly equal to zero).

- Since all of the data are defined over the field K, each point $f^{\circ n}(a)$ is K-rational so that for $n \in E$, $f^{\circ n}(a) \in Y(K)$. That is, the dynamical Mordell-Lang conjecture refines the description of the K-rational points on Y.
- If Y contains a subvariety $Z \subseteq Y$ which is periodic under the action of f, say of period c, and $f^{\circ b}(a) \in Z(K)$, then for every $m \in \mathbb{N}$ we have $b + mc \in E$. Thus, it is necessary to include arithmetic progressions in the conjecture.

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Why "Mordell-Lang"? Some history

- Mordell's original conjecture (proven by Faltings) asserts that a geometrically irreducible projective curve of genus at least two defined over the rational numbers has at most finitely many rational points. This conjecture is known to hold with $\mathbb{Q}$ replaced by any finitely generated field of characteristic zero so that *a fortiori* the dynamical Mordell-Lang conjecture holds for Y a high genus curve.
- The Mordell-Weil theorem asserts that if K is a finitely generated field and A is an abelian variety over K, then $A(K)$ is a a finitely generated commutative group. Using the embedding of a curve in its Jacobian, the Mordell conjecture may be deduced as a corollary of the Mordell-Lang conjecture (also a theorem of Faltings) that if A is an abelian variety over a field K of characteristic zero, $Y \subseteq A$ is a subvariety, and $\Gamma \leq A(K)$ is a finitely generated subgroup, then $Y(K) \cap \Gamma$ is a finite union of translates of subgroups of Γ.

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Why "Mordell-Lang"? A translation

Consider again X an abelian variey over a field K of characteristic zero, $\Gamma \leq A(K)$ a finitely generated group, and $Y \subseteq X$ a subvariety.

We may let Γ act on X via translation and take our starting point a to be 0_X. Then Faltings Theorem says that the return set is a finite union of cosets of subgroups.

Specializing further, if $\Gamma \cong \mathbb{Z}$, then because the cosets of subgroups of $\mathbb{Z}$ are exactly the (two-sided) arithmetic sequences (possibly with modulus 0), the dynamical Mordell-Lang conjecture for translation actions on abelian varieties is a special case of Faltings' Theorem.

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Failure of higher rank dynamical Mordell-Lang

Faltings' Theorem suggests that it would make sense to consider return sets for actions of more complicated semigroups than merely $\mathbb{N}$. Why not transpose Faltings' theorem to general algebraic dynamical systems given by actions of finitely generated semigroups?

- Without some restriction on the maps appearing in a higher rank algebraic dynamical system, one cannot hope for a reasonable description of the return sets. For example, if we allow $S := \mathbb{Z}^n$ to act on X being affine n-space by translation and take $a = (0, \dots, 0)$ and take Y to be any embedded affine variety in n space, then the return set is essentially $Y(\mathbb{Z})$.
- Even if one restricts to actions whose geometry is closely related to that of the classical Mordell-Lang conjecture the return sets may be arbitrarily complicated. With Yu Yasufuku, I showed that for any set $T \subseteq S = \mathbb{N}^n$ expressible as the solution set to a system of exponential-Diophantine equations, there is an algebraic dynamical system $S \curvearrowright X$ with X a Cartesian power of the multiplicative group and a point $a \in X$ and subvariety $Y \subseteq X$ so that $E(X, S, a, Y) = T$.

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Skolem-Mahler-Lech theorem: linear recurrence sequences

By a linear recurrence sequence $(u_n)_{n=0}^{\infty}$ in a field K we mean a sequence of elements of K for which there is some $d \in \mathbb{Z}_+$ and elements $\lambda_0, \ldots, \lambda_{d-1}$ of K so that for all $n \in \mathbb{N}$ we have $u_{n+d} = \sum_{j=0}^{d-1} \lambda_j u_{n+j}$.

The following are linear recurrence sequences

- $0, 1, 0, 1, 0, 1, 0, \ldots$ since $u_{n+2} = u_n$.
- $1, -5, 25, -125, 625, \ldots, (-5)^n, \ldots$ since $u_{n+1} = (-5)u_n$.
- $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$ since $u_{n+2} = u_{n+1} + u_n$
- $0, 1, 5, 19, 65, 211, \ldots, 3^n - 2^n, \ldots$ since $u_{n+2} = 5u_{n+1} - 6u_n$.
- $0, 2, 8, 24, 64, 160, \ldots, n2^n, \ldots$ since $u_{n+2} = 4u_{n+1} - 4u_n$.

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Skolem-Mahler-Lech: the theorem

Theorem (Skolem,Mahler,Lech)

Let $(u_n)_{n=0}^{\infty} \in \mathbb{C}^{\mathbb{N}}$ *be a linear recurrence sequence of complex numbers. Then* $\{n \in \mathbb{N} \ : \ u_n = 0\}$ *is a finite union of points and arithmetic progressions.*

Why do I mention this theorem here?

- Properly interpreted, it may be seen as an instance of the dynamical Mordell-Lang conjecture itself.
- Its proof generalizes to give one of the most effective techniques for proving instances of the dynamical Mordell-Lang conjecture.
- Our new theorems pass through a study of equations involving linear recurrence sequences in the style of Skolem-Mahler-Lech.

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Skolem-Mahler-Lech: geometric reformulation

Consider a linear recurrence sequence $(u_n)_{n=0}^{\infty}$ satisfying $u_{n+d} = \sum_{i=0}^{d-1} \lambda_i u_{n+i}$. Adjusting if need be, we may assume that $\lambda_0 \neq 0$.

We have the usual action $\mathrm{GL}_n \curvearrowright \mathbb{A}^n$ and inside of this action of the general linear group we may identify an action by $\mathbb{N}$ with generator $f : (x_0, \ldots, x_{d-1}) \mapsto (x_1, \ldots, x_{d-1}, \sum_{i=0}^{d-1} \lambda_i x_i)$.

Let $a := (u_0, \ldots, u_{d-1}) \in \mathbb{A}^n =: X$ and let $Y \subseteq X$ be defined by $x_0 = 0$. Then the return set for this algebraic dynamical system is precisely $\{n \in \mathbb{N} \ : \ u_n = 0\}$.

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Skolem-Mahler-Lech: proof technique

- Using basic linear algebra, one shows that the entries of the matrix for $f^{\circ n}$ may be expressed in the form $\sum P_j(n)\mu_j^n$ for suitable polynomials P_j and bases μ_j, at least for $n \gg 0$.
- It follows that (again for $n \gg 0$), u_n may also be expressed as an exponential polynomial.
- Skolem's key insight was that possibly after restricting to arithmetic subsequences, it is possible to extend the function $n \mapsto u_n$ to a p-adic analytic function $\mathbb{Z}_p \to \mathbb{Z}_p$ and such functions can have infinitely many zeros only if they vanish on some coset of $p^N\mathbb{Z}_p$.

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Successes of Skolem's method

- The strongest general theorem towards the dynamical Mordell-Lang conjecture is the theorem of Bell-Ghioca-Tucker in which they prove the conjecture for étale maps $f : X \to X$ by generalizing Skolem's method showing that (again, possibly after restricting a subsequence and choosing p appropriately) the map $n \mapsto f^{\circ n}(a)$ extends to a p-adic analytic function.
- In a paper with Benedetto, Ghioca, Hutz, Kurlberg and Tucker, we show how to use Skolem's method to prove instances of the dynamical Mordell-Lang conjecture for split algebraic dynamical systems on $(\mathbb{P}^1)^n$. We also give a probabilistic argument as to why one show expect that Skolem's method will fail in general for dynamical systems on varieties of dimension three or greater.
- We should note here the success of Xie in proving some instances of the dynamical Mordell-Lang conjecture (*e.g.* for polynomial endomorphisms of $\mathbb{A}^2$) using entirely different methods.

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Complications in positive characteristic

The naïve transposition of the dynamical Mordell-Lang conjecture fails in positive characteristic, for the same reason that the usual Mordell-Lang conjecture does.

For example, if $K = \mathbb{F}_p(t)$, $X = \mathbb{A}^2$, $a = (1,1)$, Y is defined by $x_0 + x_1 = 1$, and $f : X \to X$ is given by $(x_0, x_1) \mapsto (tx_0, (1-t)x_1)$, then for any $n \in \mathbb{N}$ we have $f^{\circ n}(a) = (t^n, (1-t)^n)$ and $f^{\circ n}(a) \in Y$ if and only if n is a power of p.

Examples with more complicated return sets may be constructed. For example, one may obtain $\{\sum_{i=1}^d c_i p^{kn_i} \ : \ (n_1, \ldots, n_d) \in \mathbb{N}^d\}$ for suitably chosen natural numbers d and k and rational numbers $c_1, \ldots, c_d$.

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One might propose the following conjecture, though given the paucity of evidence to date, it might be better to pose this as a question.

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Let K be a field of characteristic $p > 0$, X an algebraic variety over K, $f : X \to X$ a regular self-map, $a \in X(K)$ a K-rational point, and $Y \subseteq X$ a subvariety. Then the return set $\{n \in \mathbb{N} \; : \; f^{\circ n}(a) \in Y(K)\}$ is a finite union of points, arithmetic progressions and p-sets.

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With Moosa (and then improved by Ghioca and Moosa), I showed that if $\Gamma \leq X(K)$ is a finitely generated subgroup and $Y \subseteq X$ is a subvarierty, then $\Gamma \cap Y(K)$ may be expressed as a finite union of sets of the form $a + \Xi + T$ where $\Xi \leq \Gamma$ and T is an F-set.

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We work with a field K of characteristic $p > 0$ and take $X = \mathbb{G}_m^g$ for some $g \in \mathbb{Z}_+$ (that is, $X(K) = (K^\times)^g$).

It follows on general grounds that our self-map $f : X \to X$ must take the form of an endomorphism of algebraic groups followed by a translation. More concretely, $f(x_1, \ldots, x_g) = (c_1 x_1^{n_{1,1}} x_2^{n_{1,2}} \cdots x_g^{n_{1,g}}, \ldots, c_g x_1^{n_{g,1}} \cdots x_g^{n_{g,g}})$ for suitable nonzero $c_j \in K^\times$ and integers $n_{i,j}$.

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Our new theorems

We work over K a field of characteristic p, $X = \mathbb{G}_m^g$ for some $g \in \mathbb{Z}_+$, $f : X \to X$, $Y \subseteq X$ is a subvariety and $a \in X(K)$ is a K-rational point. Then:

- If $f : X \to X$ is an endomorphism and for no positive dimensional subgroup $G \leq X$ does some power of f restrict to a power of the Frobenius morphism on G, then the positive characteristic dynamical Mordell-Lang conjecture holds in this case.
- Without any condition on f, the conjecture holds when $\dim(Y) \leq 2$.
- Even with these restrictions on X, the problem for general f and Y is provably equivalent to a hard exponential-Diophantine problem over the integers.

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Reduction to exponential diophantine equations

Recall the context: K is a field of characteristic $p > 0$, $X = \mathbb{G}_m^g$, $a \in X(K)$, $f : X \to X$ and $Y \subseteq X$ is a subvariety. We write the group operation on X additively.

- We can find a finitely generated subgroup $\Gamma \leq X(K)$ with $a \in \Gamma$. (How? Let $R \subseteq K$ be a finitely generated subring over which f and a are define. Set $\Gamma := \mathbb{G}_m^g(R)$.) It is easy to reduce to the case that Γ is torsion free. Fix a basis $\gamma_1, \ldots, \gamma_r$ of Γ.
- The intersection of Y with the orbit of a is contained in the intersection $Y(K) \cap \Gamma$, is a finite union of sums of translates of groups and F-sets, for $F : X \to X$ the usual Frobenius.
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The theorem: for nondegenerate endomorphisms

Our hypothesis is that $f : X \to X$ is an endomorphism and for no positive dimensional subgroup $G \leq X$ is the restriction of some iterate of f to G a power of the Frobenius.

- A linear recurrence sequence u_n may always (at least for $n \gg 0$) be expressed as an exponential polynomial function $u_n = \sum P_i(n)\alpha_i^n$ for suitable polynomials P_i and bases α.
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Now $f : \mathbb{G}_m^g \to \mathbb{G}_m^g$ is a general self-map and $Y \subseteq \mathbb{G}_m^g$ has dimension at most two.

- In our theorem on F-sets, we see that in general the F-sets which appear in the expression for $Y \cap \Gamma$ take the form $\{\sum_{i=1}^d F^{kn_i}(a_i) \; : \; (n_1, \ldots, n_d) \in \mathbb{N}^d\}$ with $d \leq \dim(Y)$. Thus, in our case, we may take $d \leq 2$.
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Reduction of the exponential diophantine problem to dynamical Mordell-Lang

The Diophantine problem of describing $\{n \in \mathbb{N} \ : \ (\exists (m_1, \ldots, m_d) \in \mathbb{N}^d) \ u_n = \sum_{i=1}^d c_i p^{m_i}\}$ is out of reach of currently available techniques for $d \geq 3$.

By using the same coding techniques used to show that every set defined by exponential-Diophantine equations may be realized as a dynamical return set, ew show the following proposition.

Proposition

Let $T \subseteq \mathbb{N}$ be a set of natural numbers and suppose that there is some field K of characteristic $p > 0$, endomorphism $h : \mathbb{G}_m^r \to \mathbb{G}_m^r$, subvariety $Y \subseteq \mathbb{G}_m^r$ and point $a \in \mathbb{G}_m^r(K)$ for which $T = \{n \in \mathbb{N} \ : \ h^{\circ n}(a) \in Y(K)\}$. Then for any linear recurrence sequence (u_n) there is a self-map $f : \mathbb{G}_m^g \to \mathbb{G}_m^g$, point $b \in \mathbb{G}_m^g(K)$, and subvariety $W \subseteq \mathbb{G}_m^g$ so that for all $n \in \mathbb{N}$ one has $f^{\circ n}(b) \in W(K) \iff u_n \in T$.

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A class of exponential-Diophantine problems realizable as instance of dynamical Mordell-Lang

Corollary

Let p be a prime number and $c_1, \ldots, c_d \in \mathbb{Z}_+$ positive integers with $1 + \sum c_i < p$ and (u_n) a linear recurrence sequence. Then there are an integer g, an endomorphisms $f : \mathbb{G}_m^g \to \mathbb{G}_m^g$, a point $a \in \mathbb{G}_m^g(\mathbb{F}_p(t))$, and a subvariety $Y \subseteq \mathbb{G}_m^g$ for which $\{n \in \mathbb{N} \ : \ f^{\circ n}(a) \in Y(\mathbb{F}_p(t))\} = \{n \in \mathbb{N} \ : \ (\exists (m_1, \ldots, m_d) \in \mathbb{N}^d) \ u_n = \sum c_i p^{m_i}\}$.

Open problems

- While Skolem's method with p-adic uniformization does not work in positive characteristic, one might prefer to uniformize orbits by $\mathbb{F}_p[[t]]$. How far does such positive characteristic uniformization go towards the positive characteristic dynamical Mordell-Lang problem?
- What p-sets are realizable as intersections of subvarieties of tori with cyclic subgroups?
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