Sheaves on subanalytic sites

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Motivations

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Since the study of these spaces is very important, many ways have been explored by the specialists to overcome this problem.

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One of the solutions was given by Kashiwara and Schapira in 2001, with the notion of subanalytic site.

M. KASHIWARA, P. SCHAPIRA *Ind-sheaves*, Astérisque **271** (2001).

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The idea is to consider less open and less coverings (i.e. a Grothendieck topology) and extend the classical machinery of sheaves to this context.

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In this way we can apply the powerful tools of homological algebra to a larger family of functional spaces.

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That is, a contravariant functor $F : \mathrm{Op}(X) \to \mathrm{Mod}(k)$.

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A sheaf is a presheaf satisfying the following gluing conditions. Let U be open and let $\{U_j\}_{j\in J}$ be a covering of U. We have the exact sequence

$$0 \to F(U) \to \prod_{j\in J} F(U_j) \to \prod_{j,k\in J} F(U_j \cap U_k)$$

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- if $s_j \in \Gamma(U_j;F)$ such that $s_j = s_k$ on $U_j \cap U_k$ then they glue to $s \in \Gamma(U;F)$ (i.e. $s|_{U_j} = s_j$)

Examples

Let us consider

$$\begin{aligned} \mathcal{C}_X : \text{Open sets of } X &\to \mathrm{Mod}(\mathbb{R}) \\ U &\mapsto \{\text{continuous real valued functions on } U\} \end{aligned}$$

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- If s is a continuous function and s is zero on a covering of U then $s = 0$.
- If $\{s_i\}$ are continuous functions on a covering $\{U_i\}$ of U, such that $s_i = s_j$ on $U_i \cap U_j$, then there exists s continuous on U with $s = s_i$ on each U_i.

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$\Rightarrow$ The correspondence $U \mapsto \mathcal{C}_X(U)$ defines a sheaf on X

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- For example, let $X = \mathbb{R}$, $U_n = (-n, n)$, $n \in \mathbb{N}$, and $s_n \colon U_n \to \mathbb{R}$, $x \mapsto x^2$. Then s_n is bounded on U_n for each $n \in \mathbb{N}$, but $x \mapsto x^2$ is not bounded on $\mathbb{R}$.

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Let $F \in \mathrm{Mod}(k_X)$ we define the fiber of F at x as

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Moreover, given $U_1, U_2 \ni x$ and $s_i \in U_i$, we have $s_1 \equiv s_2$ in F_x if $s_1 = s_2$ on a neighborhood of x $W \subset U_1 \cap U_2$.

Fibers

Two sheaves F, G are isomorphic if

$$F_x \simeq G_x$$

for any $x \in X$. More generally a sequence of sheaves

$$0 \to F' \to F \to F'' \to 0$$

is exact if the sequence

$$0 \to F'_x \to F_x \to F''_x \to 0$$

is exact in $\mathrm{Mod}(k_X)$.

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One can generalize this notion by choosing a subfamily of open subsets $\mathcal{T}$ of X and for each U a subfamily $\mathrm{Cov}(U)$ of coverings of U satisfying suitable hypothesis (defining a site $X_{\mathcal{T}}$).

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Then $F : \mathcal{T} \to \mathrm{Mod}(k)$ is a sheaf on $X_{\mathcal{T}}$ if for each $U \in \mathcal{T}$ and each $\{U_j\}_{j \in J} \in \mathrm{Cov}(U)$ we have the exact sequence

$$0 \to F(U) \to \prod_{j \in J} F(U_j) \to \prod_{j,k \in J} F(U_j \cap U_k)$$

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- If $\{s_i\}$ are bounded on a finite covering $\{U_i\}_{i=1}^n$ of U, such that $s_i = s_j$ on $U_i \cap U_j$, then there exists s bounded on U with $s = s_i$ on each U_i.

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$\Rightarrow$ The correspondence $U \mapsto \Gamma(U; \mathcal{C}^b_X)$ defines a sheaf on $X_{\mathcal{T}}$.

The general case

Let X be a topological space and consider a family of open subsets $\mathcal{T}$ satisfying:

$$\begin{cases} \text{(i) } U, V \in \mathcal{T} \Leftrightarrow U \cap V, U \cup V \in \mathcal{T}, \\ \text{(ii) } U \setminus V \text{ has finite numbers of connected components } \forall U, V \in \mathcal{T}, \\ \text{(iii) } \mathcal{T} \text{ is a basis for the topology of } X. \end{cases}$$

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3. $\mathcal{T}$={open relatively compact subanalytic subsets of a real analytic manifold}, the subanalytic site X_{sa}.
4. $\mathcal{T}$={open definable subsets of N^n}, given an O-minimal structure $(N, <, \dots)$, the site X_{def}.

Construction of sheaves on $X_{\mathcal{T}}$

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Then F is a sheaf on $X_{\mathcal{T}}$.

Subanalytic sheaves

From now on we will consider the subanalytic site X_{sa}.

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Tempered functions

Let X be a real analytic manifold manifold and let $U \subset X$ be a relatively compact subanalytic open subset, f $\mathcal{C}^\infty$ on U is tempered if $\exists\, M, C > 0$ such that

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This is not a sheaf with the usual topology.
For example, let $X = \mathbb{R}$, then $e^{1/x}$ is tempered (even bounded) on $U_n = \{1/n < x < 1\}_{n \in \mathbb{N}}$ but it is not tempered on $\cup U_n = \{0 < x < 1\}$.

Tempered functions

Anyway one can show that if U, V are open subanalytic the sequence

$$0 \to \mathcal{C}_X^{\infty,t}(U \cup V) \to \mathcal{C}_X^{\infty,t}(U) \oplus \mathcal{C}_X^{\infty,t}(V) \to \mathcal{C}_X^{\infty,t}(U \cap V)$$

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is exact. This implies that $U \mapsto \mathcal{C}_X^{\infty,t}(U)$ is a sheaf on the subanalytic site X_{sa}.

Fibers

In the case of subanalytic sheaves we do not have the notion of fibers in the usual sense, i.e. if we consider

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i.e. there are $F \not\simeq G$ even if $F_x \simeq G_x$ $\forall x \in X$.

Example: Let $X = \mathbb{R}$ and consider the sheaves $\mathcal{C}_{\mathbb{R}}$ and $\mathcal{C}^b_{\mathbb{R}}$. Then $\mathcal{C}_{\mathbb{R},x} \simeq \mathcal{C}^b_{\mathbb{R},x}$ $\forall x \in \mathbb{R}$. Indeed, any continuous function f in $(x - \varepsilon, x + \varepsilon)$, $\varepsilon > 0$ is bounded in $(x - \varepsilon/2, x + \varepsilon/2)$.

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We need to consider more points.

Spectral topology

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A neighborhood of an ultrafilter α is a globally subanalitic open subset U contained in α.
We call $\widetilde{X}$ the associated topological space. In $\widetilde{X}$ any covering of a relatively compact subanalytic open subset has a finite subcover.

Example

For example, the points of $\widetilde{\mathbb{R}}$ are the following. Let $x \in \mathbb{R}$

1. $\{S \,\text{subanalytic}, \, S \supseteq x\}$ (the point x)
2. $\{S \,\text{subanalytic}, \, S \supseteq (x, x+\varepsilon), \ \varepsilon > 0\}$ (the point x^+)
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Thanks to these new points we can distinguish $\mathcal{C}_{\mathbb{R}}$ from $\mathcal{C}^b_{\mathbb{R}}$ on $\widetilde{\mathbb{R}}$. For example let $f(x) = x^{-1}$. Then $f \notin \mathcal{C}^b_{\mathbb{R}}(0, \varepsilon)$ $\forall \varepsilon > 0$.

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Topological and subanalytic sheaves

Theorem:

Let X be a real analytic manifold. The categories $\mathrm{Mod}(k_{X_{sa}})$ and $\mathrm{Mod}(k_{\widetilde{X}})$ are equivalent.

Hence, if we want to work on fibers on X_{sa}, we have to consider the topological space $\widetilde{X}$.

Operations

Theorem:

Let $f : X \to Y$ be a morphism of real analytic manifolds. The six Grothendieck operations $\mathcal{H}om$, $\otimes$, f_*, f^{-1}, $f_{!!}$, $f^!$ are well defined for subanalytic sheaves.

L. PRELLI *Sheaves on subanalytic sites*, Rendiconti del Seminario Matematico dell'Università di Padova Vol. **120** (2008).

Generalizations (joint work with M. Edmundo)

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1. (regular) definable spaces in o-minimal expansions of real closed fields;
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3. definably compact groups in an arbitrary o-minimal structure

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The Grothendieck formalism allows us to obtain o-minimal versions of:

1. derived projection formula;
2. derived base change formula;
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The ring of differential operators

Let X be a complex analytic manifold. We denote by $\mathcal{D}_X$ the sheaf of rings of differential operators. Locally, a section of $\Gamma(U; \mathcal{D}_X)$ may be written as $P = \sum_{|\alpha| \leq m} a_\alpha(z) \partial_z^\alpha$ with $a_\alpha(z)$ holomorphic on U.
We denote by $\mathrm{Mod}(\mathcal{D}_X)$ the sheaf of $\mathcal{D}_X$-modules.

Complex of solutions

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Definition: If U is open, $\mathcal{F}$ a $\mathcal{D}_X$-module, P a differential operator, $\mathcal{S}ol_{\mathcal{F}}(P)$ on U is the complex

$$\Gamma(U;\mathcal{F}) \stackrel{P}{\to} \Gamma(U;\mathcal{F})$$

$$\begin{aligned} H^0(U;\mathcal{S}ol_{\mathcal{F}}(P)) &= \{s \in \Gamma(U;\mathcal{F}),\ Ps = 0\} = \ker P \\ H^1(U;\mathcal{S}ol_{\mathcal{F}}(P)) &= \Gamma(U;\mathcal{F})/P\Gamma(U;\mathcal{F}) = \mathrm{coker} P \end{aligned}$$

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Definition: P_1 and P_2 are equivalent if for any $\mathcal{F}$ $\ker P_1 \simeq \ker P_2$ and $\operatorname{coker} P_1 \simeq \operatorname{coker} P_2$ (i.e. $\mathcal{S}ol_{\mathcal{F}}(P_1)$ and $\mathcal{S}ol_{\mathcal{F}}(P_2)$ are quasi-isomorphic).

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If $\alpha \neq -1$, one can verify that the morphisms

$$\mathcal{S}ol_{\mathcal{F}}(P_\alpha) \underset{\frac{\partial_z}{\alpha+1}\cdot}{\overset{z\cdot}{\rightleftarrows}} \mathcal{S}ol_{\mathcal{F}}(P_{\alpha+1})$$

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$$P_\alpha = z\partial_Z - \alpha \quad P_{\alpha+1} = z\partial_z - \alpha - 1.$$

If $\alpha \neq -1$, one can verify that the morphisms

$$\mathcal{S}ol_{\mathcal{F}}(P_\alpha) \underset{\frac{\partial z}{\alpha+1}\cdot}{\overset{z\cdot}{\rightleftarrows}} \mathcal{S}ol_{\mathcal{F}}(P_{\alpha+1})$$

induce a quasi- isomorphism between the complexes $\mathcal{S}ol_{\mathcal{F}}(P_\alpha)$ and $\mathcal{S}ol_{\mathcal{F}}(P_{\alpha+1})$.

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Hence P_α and $P_{\alpha+1}$ are equivalent.

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Hence $z(z\partial_z + 1)$ and $z^2\partial z + 1$ are not equivalent (even if the holomorphic solutions are).

Equivalence for regular operators (dim 1)

Definition: $P = \sum_{\alpha \leq n} a_\alpha(z)\partial^\alpha$, $a_n(z) \neq 0$ when $z \neq 0$, is regular at 0 if for each $j \leq n$, $n - \mathrm{ord}_0(a_n) \geq j - \mathrm{ord}_0(a_j)$.

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1. P and Q are equivalent.
2. The kernels and cokernels of

$$\mathcal{O}_X \xrightarrow{P} \mathcal{O}_X \quad \mathcal{O}_X \xrightarrow{Q} \mathcal{O}_X$$

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are isomorphic (i.e. $\mathcal{S}ol_{\mathcal{O}_X}(P)$ is quasi-isomorphic to $\mathcal{S}ol_{\mathcal{O}_X}(Q)$). In particular the holomorphic solutions are sufficient to establish if two regular equations are equivalent.

Subanalytic sheaves and solutions

The subanalytic sheaf $\mathcal{O}^t_X$ of tempered holomorphic functions has a structure of $\rho_!\mathcal{D}_X$-module. ($\Gamma(U; \rho_!\mathcal{D}_X)$ are differential operators $\sum_{|\alpha|\leq m} a_\alpha \partial_z^\alpha$ with a_α holomorphic in $\overline{U}$)

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Let us consider the irregular operators $z^2\partial_z + 1$ and $z^3\partial_z + 2$. Their solutions are $\exp(z^{-1})$ and $\exp(z^{-2})$.

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Theorem: There exists an open subanalytic U such that $\exp(z^{-1}) \in \mathcal{O}^t_X(U)$ and $\exp(z^{-2}) \notin \mathcal{O}^t_X(U)$.

In particular for such U we have

$$\begin{aligned} H^0(U; \mathcal{S}ol_{\mathcal{O}^t_X}(z^2\partial_z + 1)) &\simeq \mathbb{C} \cdot \exp(z^{-1}) \\ H^0(U; \mathcal{S}ol_{\mathcal{O}^t_X}(z^3\partial_z + 2)) &\simeq 0. \end{aligned}$$

Irregular differential operators

In general

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Hence thanks to tempered holomorphic solutions we can distinguish irregular differential operators which cannot be distinguished with holomorphic solutions.

Recent developments on this subject have been performed by D'Agnolo-Kashiwara and Schapira-Guillermou.

Functorial constructions

Let $X = \mathbb{C}$. On the normal bundle T_0X it is possible to define a functor (specialization)

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such that $(\nu_0 F)_\xi = \varinjlim_{S \ni \xi} F(S)$, S sector.
(This construction can be performed in general for a closed submanifold of an analytic manifold)

M. KASHIWARA, P. SCHAPIRA *Sheaves on Manifolds*, Grundlehren der Math. **292**, Springer-Verlag, Berlin (1990).

Functorial constructions

This construction can be extended to subanalytic sheaves and when $F = \mathcal{O}_X^{\mathrm{w}}$ of Whitney holomorphic functions we have

$$\nu_0 \mathcal{O}_X^{\mathrm{w}} \simeq \mathcal{A}_X$$

where $\mathcal{A}_X$ is the sheaf (on the blow-up at 0) of holomorphic functions asymptotically developable at the origin.

L. PRELLI *Microlocalization of subanalytic sheaves*, Mémoires de la SMF (2013).

Functorial constructions (joint work with N. Honda)

It is possible to construct a specialization functor ν_D with respect to a normal crossing divisor D in $\mathbb{C}^n$.

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It is possible to construct a specialization functor ν_D with respect to a normal crossing divisor D in $\mathbb{C}^n$.

Applying this functor to the sheaf of Whitney holomorphic functions we obtain Majima's sheaf $\mathcal{A}$ of strongly asymptotically developable holomorphic functions.

N. HONDA, L. PRELLI *Multi-specialization and multi-asymptotic expansions*, Advances in Math. **232**, pp. 432-498 (2013).

Sheaves on subanalytic sites

Luca Prelli

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