Homotopy of groups definable in o-minimal structures

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O-MINIMAL STRUCTURES

A structure (expanding a real closed field) R is a collection $\mathcal{S} = (\mathcal{S}_n)_{n\in\mathbb{N}}$, with $\mathcal{S}_n \subset R^n$ s.th. for each $n \geq 0$

(1) $\{(x_1, \dots, x_n) \in R^n \colon P(x_1, \dots, x_n) > 0\} \in \mathcal{S}_n$,
f. e. $P(T_1, \dots, T_n) \in \mathbb{Z}[T_1, \dots, T_n]$.

(2) $\mathcal{S}_n$ is a Boolean algebra.

(3) If $X \in \mathcal{S}_n$ and $Y \in \mathcal{S}_m$ then $X \times Y \in \mathcal{S}_{n+m}$.

(4) If $p : R^{n+1} \to R^n$ is the projection and $X \in \mathcal{S}_{n+1}$ then $p(X) \in \mathcal{S}_n$.

Given a structure $(R, \mathcal{S})$, the elements of the $\mathcal{S}_n$'s are called sets definable without parameters.

A subset $X \subset R^n$ is definable if there are $c_1, \dots, c_m \in R$ and $S \in \mathcal{S}_{n+m}$ s.th.

$$(x_1, \dots, x_n) \in X \Longleftrightarrow (c_1, \dots, c_m, x_1, \dots, x_n) \in S.$$

A structure $(R, \mathcal{S})$ is o-minimal if also satisfies:

(5) The $X \subset R$ definable are a finite union of intervals (with end points in $R \cup \{\pm\infty\}$).

FACT. The property of being o-minimal depends only on $\mathcal{S}$ (not on R).

Given a structure $(R, \mathcal{S})$, an $f\colon X \to R^m$, with $X \subset R^n$ is a definable function if its graph is definable.
All maps are assume to be continuous.

$$\langle R, \{f_i\}_{i\in I}, \{P_j\}_{j\in J}\rangle$$

denotes the structure $(R, \mathcal{S})$, where $\mathcal{S}$ is the minimal collection containing the functions f_i's, and the sets P_j's.

EXAMPLES OF O-MINIMAL STRUCTURES

• [Tarski'51] R a real closed field (definable = semialgebraic).

• [Łojasiewicz'65-Gabrielov'68-v.d.Dries 86]

$$\mathbb{R}_{an} = \langle \mathbb{R}, \{f : [-1,1] \to \mathbb{R}\}_{f\in\mathcal{F}}\rangle,$$

where $\mathcal{F}$ is a set of real analytic functions.

• [Wilkie'96] $\mathbb{R}_{exp} = \langle \mathbb{R}, exp\rangle$.

• [Lipshitz & Robinson'06]
Let $R = \{\sum_{i\geq m} a_i t^{i/q}\colon m \in \mathbb{Z}, q \in \mathbb{N}, a_i \in \mathbb{R}\}$.

$$p(t) > 0 \iff a_m > 0$$

(t is infinitesimal).

Each $s = s(z_1, \ldots, z_n) \in \mathbb{R}[[z_1, \ldots, z_n]]$ determines

$$f_s\colon [-t,t]^n \to \mathbb{R}\colon (a_1, \ldots, a_n) \mapsto s(a_1, \ldots, a_n)$$

(the series s converges in the infinitesimal cube).

$$\langle R, \{f_s\}_{s\in\mathbb{R}[[z_1,\ldots,z_n]]}\rangle.$$

CONSEQUENCES OF O-MINIMALITY

Definable choice. Let X be a definable set and $\equiv$ a definable equivalence relation on X. Then $X/\equiv$ is a definable set.

Curve selection. Let $X \overset{def}{\subset} R^n$, $x_0 \in cl(X)$. Then there is a definable map $\gamma\colon [0,1) \to R^n$ s.th. $\gamma(0) = x_0$ and $\gamma(0,1) \subset X$.

Definably compactness. A definable set X is definably compact if it satisfies one of the following equivalent properties:
(1) X is closed and bounded.
(2) Every definable map $(0,1) \to X$ extends (by continuity) to a map $[0,1) \to X$.

Definably connectedness. A definable set X is definably connected if it satisfies one of the following equivalent properties:
(1) X is not the union of two disjoint nonempty definable open subsets of X.
(2) $\forall x, y \in X\ \exists \gamma\colon [0,1] \to X$ definable map such that $\gamma(0) = x$ and $\gamma(1) = y$.
FACT. A definable set has a finite number of definably connected components.

Triangulation theorem. Let $X \overset{def}{\subset} R^n$ be definably compact. Then, there is a finite simplical complex K with vertices in $\mathbb{Q}^n$ and a definable homeomorphism

$$\Phi\colon |K|(R) \to X$$

where $|K|(R)$ is the realization of K in R.

Dimension. Define $dim(X) := dim(|K|)$.

TRANSFER RESULTS

For $X \overset{def}{\subset} R^n$ definably compact, identify X with $|K|(R)(\overset{def}{\approx} X)$ We can also consider $|K|(\mathbb{R}) \subset \mathbb{R}^n$ and compare them.

$$\#\pi_0(|K|(R)) = \#\pi_0(|K|(\mathbb{R})) \tag{1}$$

where the o-minimal π_0 (lhs) is the set of definable connected components.

FACT. [Berarducci & Ot.'02]

$$\pi_1(|K|(R)) \cong \pi_1(|K|(\mathbb{R})) \tag{2}$$

where the o-minimal fundamental group (lhs) is based on definable loops and definable homotopies between them.

$$H_k(|K|(R)) \cong H_k(|K|(\mathbb{R})) \quad \text{for all } k \geq 0 \tag{3}$$

where o-minimal singular homology (lhs) is based on definable maps $\Delta_k(R) \to |K|(R)$, Δ_k the standard k-dimensional simplex.

Proof. (2) The o-minimal fundamental group admits a combinatorial definition in terms of the finite simplicial complex K.
(3) Both singular homology groups are isomophic to the corresponding simplicial homology groups of K. The classical proof uses only the Eilenberg-Steenrod axioms and we have it also in the o-minimal context thanks to [Woerheide'96]. □

The topological rhs of (1), (2) and (3) depends only on K.

HOMOTOPY TRANSFER

Let $X(R)$ and $Y(R)$ be semialgebraic sets, definably compact and defined without parameters.

THM. [Baro & Ot.Preprint'08] **(1)** For every definable $f\colon X(R) \to Y(R)$, there is a $g\colon X(R) \to Y(R)$ semialgebraic defined w/out parameters s.th. $f \overset{def}{\sim} g$.

(2) Let $f, g\colon X(R) \to Y(R)$ be semialgebraic defined w/out parameters. If $f \overset{def}{\sim} g$ then there is a semialgebraic homotopy H between f and g, with H defined w/out parameters.

In particular, there is a natural bijection:

$$\{f\colon X(R) \to Y(R) \mid f \text{ s.a.}\}/_{\underset{\sim}{s.a.}} \to \{f\colon X(R) \to Y(R) \mid f \text{ def}\}/_{\underset{\sim}{def}}$$

$$[f] \longrightarrow [f]$$

Proof. Based on the normal triang.thm [Baro,Preprint'07].□

DEF. Let Z be a definable set and $z_0 \in Z$.
The o-minimal n-th homotopy group is
$\pi_n(Z, z_0) := \{f\colon (I^n, \partial I^n) \to (Z, z_0) \mid f \text{ definable }\}/\overset{def}{\sim}$,
for each $n > 0$.

COR. $\pi_n(X(R)) \cong \pi_n(X(\mathbb{R}))$, for every $n > 0$
(lhs: o-minimal, rhs: topological).
Proof. Let $\bar{\mathbb{Q}}$ be the field of real algebraic numbers. By the thm $\pi_n(X(R)) \cong \pi_n(X(\bar{\mathbb{Q}}))$ (lhs: o-min, rhs: s.a.).
By [Delfs & Knebusch'85] $\pi_n(X(\bar{\mathbb{Q}})) \cong \pi_n(X(\mathbb{R}))$. □

DEFINABLE GROUPS

DEF. A group G is definable if both $G \subset R^m$ and the graph of the group operation are definable.

FACT.[Pillay'88] Let G be a definable group. G is equipped with a definable manifold topology making "$\cdot$" and $-^{-1}$' continuous.

EXAMPLES.
1) $(R, +)$
2) $(R^*, \cdot)$
3) $([0,1)(R), +(mod\, 1))$
4) $\mathbb{T}^d(R) = [0,1)^d(R)$
5) R-rational points of an algebraic group
6) A compact Lie group if $R = \mathbb{R}$.

WLOG.[Robson'83] The topology of G is induced by that of the ambient space R^l.

BASIC PROPERTIES.[Pillay'88] G a definable group.
- If X is a definable *large* subset of G $(dim(G \setminus X) < dim\, G)$ then finitely many translates of X cover G.
- Definable subgroups are closed.
- G has the DCC on definable subgroups.
- G is definably connected iff G has not definable subgroups of finite index.

CAUTION.[Peterzil & Steinhorn'99] There are definable groups (definably compact and abelian) of dimension > 1 without definable subgroups of dimension one.

Definably compact groups I

DEF. R is sufficiently saturated if there is a cardinal κ, bigger than the cardinality of all the paramenter sets we are going to consider, such that for any set J of cardinality less than κ, for any collection $\{X_j : j \in J\}$ of definable sets, with the finite intersection property, we have $\bigcap_{j\in J} X_j \neq \emptyset$.

R is sufficently saturated. (Small means of cardinality less than the corresponding κ.)

DEF. A $Y \subset R^m$ is type-definable if $Y = \bigcap_{j\in J} Y_j$, with the Y_j's definable and J small.

Pillay's Conjecture.[Hrushovski & Peterzil & Pillay'08] [Berarducci, Dolich, Edmundo, Starchenko and Ot.]
Let G be a definably connected definably compact group. Then,
there is a type-definable normal $H < G$ of small index s.th.

(i) G/H with the *logic topology* is a compact Lie group ($\pi : G \to G/H$, $Y \subset G/H$ closed $:\Leftrightarrow \pi^{-1}(Y)$ is type-definable)

and

(ii) $dim_{o-min}(G) = dim_{Lie}(G/H)$.

FACT. H is the smallest type-definable subgroup of G of small index. H is divisible and torsion-free.

Definably compact groups II

$$1 \to H \to G \to L_G \to 1$$

G definably connected definably compact group;
H (the unique) type-definable, divisible and torsion-free;
L_G compact Lie group, and $dim\, G = dim\, L_G$.

FACT.[Peterzil & Starchenko'00]
If G is definably compact then $G/Z(G)$ is *semisimple* (=no infinite abelian (definable) subgroups).

FACT.[Peterzil & Pillay & Starchenko'02] $_{def}$
If G is semisimple and $Z(G) = 1$ then $G \overset{def}{\cong} G_1(R)$ for some G_1 semialgebraic and defined w/out parameters.

BASIC EXAMPLES.

- If $G = \mathbb{T}^d(R)$ then $L_G = \mathbb{T}^d(\mathbb{R})$, π is the standard part map and $ker(\pi)$ is the set of infinitesimals in $\mathbb{T}^d(R)$
- If G is semisimple and $Z(G) = 1$ ($G = G(R)$) then $L_G = G(\mathbb{R})$, π is the standard part map and $ker(\pi)$ is the set of infinitesimals in $G(R)$.

OUTLINE OF PROOFS. G definably compact & definably connected.
Case 1. G abelian.
Case 2. G semisimple
Case 3. From $G/Z(G)$ and $Z(G)$ to G.

FACT.[Hrushovski & Peterzil & Pillay, Preprint Nov'08]
G definably compact & definably connected.
(1) $G \cong (S \times A)/F$ definably, with $S, A < G$ definable, S semisimple, A abelian and F finite central.
(2) $(G, \cdot) \equiv (L_G, \cdot)$.

HOMOTOPY OF DEF. COMPACT GROUPS

Motivation.
(Steps on the proof of Pillays' conjecture for G abelian.) Once you know that G/H is a compact connected Lie group and H is divisible and torsion-free, you also know that G and G/H have the same torsion subgroups:

$$\pi : G \to G/H \quad \text{induces} \quad G[k] \cong (G/H)[k].$$

Since G/H is an abelian compact Lie group, it is a torus and $(G/H)[k] \cong (\mathbb{Z}/k\mathbb{Z})^d$, where $d = dim_{Lie} G/H$. Hence, to get

$$dim_{Lie} G/H = dim_{o-min} G,$$

suffices the following.

FACT. [Edmundo-Ot.'04] Let G be a definably compact definably connected abelian group of dimension d. Then

$$G[k] \cong (\mathbb{Z}/k\mathbb{Z})^d \text{ for each } k \geq 2.$$

Proof. Based on the fact that $\pi_1(G) \cong \mathbb{Z}^d$. □

Now, it the light of Pillay's conjecture the fact that $\pi_1(G) \cong \mathbb{Z}^d$ can be expressed as:

FACT. $\pi_1(G) \cong \pi_1(L_G)$, for G abelian.

Proof. $L_G = \mathbb{T}^d \Rightarrow \pi_1(L_G) \cong \pi_1(\mathbb{S}^1) \times \overset{d}{\cdots} \times \pi_1(\mathbb{S}^1) \cong \mathbb{Z}^d$. □

QUESTION 1. Is $\pi_1(G) \cong \pi_1(L_G)$ also true in the non abelian case?
QUESTION 2. Is $\pi_n(G) \cong \pi_n(L_G)$ for all $n > 0$?
(lhs: o-minimal homotopy, rhs: homotopy)

SETTING.

$$1 \to H \to G \to L_G \to 1$$

G definably connected definably compact group;
L_G compact Lie group, and $dim\, G = dim\, L_G$.

THM 1.[Berarducci & Mamino & Ot.] $\pi_1(G) \cong \pi_1(L_G)$.

Proof. By the o-minimal Poincaré-Hurewicz theorem [Edmundo-Ot'04], and the corresponding classical result, STP that

$$H_1(G) \cong H_1(L_G) \tag{4}$$

(lhs: o-min. sing. homology, rhs: sing. homology).
By [Berarducci'07], for any abelian coefficient group A

$$H^1(G; A) \cong H^1(L_G; A) \tag{5}$$

(lhs: o-min. sing. cohomology with coefficients in A, rhs: singular cohomology with coefficients in A).
By the universal coefficient theorem for cohomology

$$H^1(G, A) \cong Hom(H_1(G), A),$$

since $H_0(G)(\cong \mathbb{Z})$ is free. By the corresponding classical result and (5), we get

$$Hom(H_1(G); A) \cong Hom(H_1(L_G); A)$$

for any abelian group A. Hence (4) follows since both groups $H_1(G)$ and $H_1(L_G)$ are abelian and finitely generated. □

SETTING.

$$1 \to H \to G \to L_G \to 1$$

G definably connected definably compact group; L_G compact Lie group, and $dim\, G = dim\, L_G$.

THM 2.[Berarducci & Mamino & Ot.] $\pi_n(G) \cong \pi_n(L_G)$, for G abelian and $n > 1$.

Proof. Since $L_G = \mathbb{T}^d$, we have

$$\pi_n(L_G) \cong \pi_n(\mathbb{S}^1) \times \overset{d}{\cdots} \times \pi_n(\mathbb{S}^1) = 0.$$

So, we need to prove that $\pi_n(G) = 0$.

Claim 1. If $\pi_n(G)$ is a finitely generated abelian group then $\pi_n(G) = 0$.

Proof of claim 1. STP $\pi_n(G)$ is divisible. Let $k > 0$.

$$p_k : G \to G : x \mapsto kx$$

is a definable covering map. Hence, p_k induces an isomorphism on the o-minimal higher homotopy groups and

$$(p_k)_* : \pi_n(G) \to \pi_n(G) : [\gamma] \to k[\gamma].$$

□

Hence, STP:

Claim 2. $\pi_n(G)$ is a finitely generated abelian group.

Claim 2. $\pi_n(G)$ is a finitely generated abelian group.

A (definable) H-space is a pointed (definable) space (X, x_0) equipped with a (definable) map

$$\mu\colon X \times X \to X$$

s. th. both maps $\mu(-, x_0)$ and $\mu(x_0, -)$ are (definably) homotopic to id_X.
Any definable group is a definable H-space. A definable H-space defined over $\mathbb{R}$ is indeed an H-space.

Proof claim 2.By the triangulation theorem $G = |K|(R)$, where K is a finite simplicial complex with rational vertexes, one of which is the group identity e. Let m be the multiplication on $|K|(R)$. So we have map

$$m : |K|(R) \times |K|(R) \to |K|(R)$$

which gives to the s.a. space defined w/out parameters $(|K|(R), e)$, the structure of a *definable* H-space.

By a previous fact, m is definably homotopic to some *semialgebraic*

$$\mu : |K|(R) \times |K|(R) \to |K|(R)$$

defined w/out parameters. Also (by the same fact) $\mu(-, x_0)$ and $\mu(x_0, -)$ are homotopic to id_X via a semialgebraic homotopy defined w/out parameters. This gives to $(|K|(R), e)$, the structure of a semialgebraic H-space with all maps defined w/out parameters.

(Cont. proof of claim 2: $\pi_n(G)$ is f.g. abelian group.
Setting: $G = |K|(R)$, $(|K|(R), e; \mu)$ a s.a. H-space with all maps defined w/out parameters.)

Transfer to the reals:

Let $|K|(\mathbb{R})$ be the realization of K in $\mathbb{R}$.

Subclaim. $\pi_n(|K|(\mathbb{R}))$ is a fin. gen. abelian group.

FACT.[Serre'53] If X is a simple space and $H_n(X)$ is finitely generated for all $n > 0$ then $\pi_n(X)$ is finitely generated for all $n > 0$.

Proof of subclaim. By transfer, the realization $\mu(\mathbb{R})$ of μ in $\mathbb{R}$ gives to $|K|(\mathbb{R})$ the structure of H-space. $|K|(R)$ definably connected implies $|K|(\mathbb{R})$ path-connected. In a path-connected H-space, the fundamental group acts trivially on the π_n's, that is, it is a simple space. Since K is a finite simplicial complex, the $H_n(|K|(\mathbb{R}))$'s are finitely generated, and so we can apply the last fact to get the subclaim. $\square$

To finish the proof of claim 2 we transfer back the results over $\mathbb{R}$ to our o-minimal structure via a previous fact: $\pi_n(|K|(\mathbb{R})) \cong \pi_n(|K|(R))$. $\square$

With this we also finish the proof of

$$\pi_n(G) \cong \pi_n(L_G) \text{ for all } n > 0,$$

in the abelian case.

COROLLARIES.

Let $\mathbb{T}^d(R) = [0,1)^d(R)$ with addition modulo 1.

COR 1. Let G be a definably connected definably compact d-dimensional abelian group. Then G is definably homotopy equivalent to $\mathbb{T}^d(R)$.
And hence $H_n(G) \cong \mathbb{Z}^{\binom{d}{n}}$, for each $0 < n \leq d$.

Proof. Consider the following map

$$\begin{array}{cccc} f: & \mathbb{T}^d(R) & \longrightarrow & G \\ & (t_1, \ldots, t_n) & \mapsto & \gamma_1(t_1) + \ldots + \gamma_n(t_d) \end{array}$$

where $\pi_1(G)$ is the abelian group freely generated by $[\gamma_1], \ldots, [\gamma_d]$. Then, by the above

$$f_*\colon \pi_n(\mathbb{T}^d(R)) \cong \pi_n(G) \text{ for } n > 0.$$

By the o-minimal version of Whitehead theorem, f is an homotopy equivalence.
Also,

$$H_n(G) \cong H_n(\mathbb{T}^d(R)) \cong H_n(\mathbb{T}^d(\mathbb{R})) \cong \mathbb{Z}^{\binom{d}{n}}$$

since homology transfers. □

COR 2. Let G be a definably connected definably compact abelian group. Then, the universal covering group of G is contractible (in the category of locally definable spaces).

THM 3.[Berarducci & Mamino & Ot.] $\pi_n(G) \cong \pi_n(L_G)$, for all $n > 1$.

Consider the functor

$$F : G \to L_G$$

from the category of definably compact groups and definable homomorphism to the category of compact Lie groups and (analytic) homomorphisms.
Pillay's conjecture says that F preserves dimension (each dim in its category). We have shown that F preserves the fundamental group.
We need to show that F preserves the higher homotopy groups.

FACT.[Berarducci'07] F is an exact functor. I.e., If $1 \to G_1 \to G_2 \to G_3 \to 1$ is an exact sequence of definably compact groups then $1 \to L_{G_1} \to L_{G_2} \to L_{G_3} \to 1$ is an exact sequence of compact Lie groups.

LEMMA 1. Let G and H be two definably connected definably compact groups. Suppose G is a finite extension of H, Then, for every $n > 1$,

$$\pi_n(H) \cong \pi_n(L_H) \Longleftrightarrow \pi_n(G) \cong \pi_n(L_G).$$

Proof. The onto homomorphism $G \to H$ with finite kernel is a definable covering map. Then, $\pi_n(G) \cong \pi_n(H)$ for any $n > 1$. By preservation of exactness, the induced map from L_G to L_H is also an onto homomorphism with finite kernel, hence a covering map and so $\pi_n(L_G) \cong \pi_n(L_H)$, for any $n > 1$. $\square$

LEMMA 2. Let G be a definably connected definably compact semisimple group.
Then, for every $n > 1$, $\pi_n(G) \cong \pi_n(L_G)$.

Proof. Since $Z(G)$ is finite, by lem. 1 WMA $Z(G) = 1$. Then, by previous facts WMA

$$G = G(M)$$

a semialgebraic group defined w/out parameters and

$$L_G = G(\mathbb{R}).$$

Then,

$$\pi_n(G(M)) \cong \pi_n(G(\mathbb{R}))$$

by transfer of homotopy. □

LEMMA 3. Let H be a definable subgroup of a definable group G. Then

$$p\colon G \to G/H$$

is a definable fibration (with the quotient topology on G/H). And hence, for each $n \geq 2$, the o-minimal homotopy groups

$$\pi_n(G, Z) \cong \pi_n(G/Z).$$

Proof. First you prove that p is a definable fibre bundle and then that every definable fibre bundle is a fibration (you have to avoid the use of path-spaces and compact-open topology). □

Proof of THM 3 ($\pi_n(G) \cong \pi_n(L_G)$, for all $n > 1$).
Let Z be the centre of G. Then, by previous fact the group G/Z is definably semisimple.

By lemma 3, for each $n \geq 2$,

$$\pi_n(G, Z) \cong \pi_n(G/Z)$$

and, hence, the o-minimal homotopy sequence of the pair (G, Z) is the following long exact sequence

$$\cdots \to \pi_{n+1}(G/Z) \to \pi_n(Z) \to \pi_n(G) \to \pi_n(G/Z) \to \pi_{n-1}(Z) \to \cdots$$

By the exactness of the functor to the Lie category we have that L_Z is a closed normal subgroup of L_G, so the projection map $L_G \to L_G/L_Z (\cong L_{G/Z})$ is a fibration and hence we have the sequence

$$\cdots \to \pi_{n+1}(L_{G/Z}) \to \pi_n(L_Z) \to \pi_n(L_G) \to \pi_n(L_{G/Z}) \to \pi_{n-1}(L_Z) \to \cdots$$

Since Z and L_Z are abelian, we have

$$\pi_n(Z) = \pi_n(L_Z) = 0 \text{ for all } n \geq 2.$$

So for $n \geq 3$, $\pi_n(G) \cong \pi_n(G/Z)$ and $\pi_n(L_G) \cong \pi_n(L_{G/Z})$.
Hence, by lemma 2

$$\pi_n(L_G) \cong \pi_n(G).$$

For $n = 2$, recall that the second homotopy group of a compact Lie group is trivial, so

$$\pi_2(L_G) = \pi_2(L_{G/Z}) = 0,$$

and therefore also $\pi_2(G/Z) = 0$ (by lemma 2). Since $\pi_2(Z) = 0$, it follows that also $\pi_2(G) = 0$. □

Merci beaucoup pour votre attention