Complex Cellular Structures

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Real cells

In tame geometry, the notion of a *cell* (also called cylindrical cell) is defined inductively as follows:

- A cell $\mathcal{C} \subset \mathbb{R}^1$ of length one is a point or an interval.
- A cell $\mathcal{C} \subset \mathbb{R}^{\ell+1}$ of length $\ell + 1$ is a set of the form

$$\mathcal{C} = \mathcal{C}_{1..\ell} \odot \mathcal{F} := \{\mathbf{x}_{1..\ell+1} : \mathbf{x}_{1..\ell} \in \mathcal{C}_{1..\ell},\ \mathbf{x}_{\ell+1} \in \mathcal{F}(\mathbf{x}_{1..\ell})\}$$

  where $\mathcal{C}_{1..\ell}$ is a cell of length ℓ and the set of fibers $\mathcal{F}$ is a family of cells of length 1:

$$\mathcal{F}(\mathbf{x}_{1..\ell}) = \{f(\mathbf{x}_{1..\ell})\} \qquad \text{or} \qquad \mathcal{F}(\mathbf{x}_{1..\ell}) = (f_1(\mathbf{x}_{1..\ell}), f_2(\mathbf{x}_{1..\ell}))$$

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The definition depends on the choice of coordinates in $\mathbb{R}^\ell$: a cell $\mathcal{C} \subset \mathbb{R}^\ell$ remains a cell after cellular maps

$$\mathbf{x} \mapsto \mathbf{y} = f(\mathbf{x}), \quad y_i = f_i(x_1, ..., x_i), i = 1, .., \ell,$$

with f_i continuous and strictly monotone in x_i for every $x_{1..i-1} \in \mathcal{C}_{1..i-1}$.

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Definition

A cell decomposition (C.D.) of a set $X \subset \mathbb{R}^\ell$ is a covering $X = \bigcup_\alpha \mathcal{C}_\alpha$ by (pairwise disjoint) cells $\mathcal{C}_\alpha \subset X$.

Cell Decompositions

Theorem (Cell Decomposition)

Theorem: Every semialgebraic set X admits a finite cell decomposition.

Denote by $\pi_{1..k}(\mathbf{x}_{1..\ell}) = \mathbf{x}_{1..k}$ the projection $\pi_{1..k} : \mathbb{R}^\ell \to \mathbb{R}^k$ onto the first k coordinates.

- Projections: C.D. of $X \implies$ C.D. of $\pi_{1..k}(X)$.
- Fibers: C.D. of $X \implies$ C.D. of $\pi^{-1}_{1..k}(p) \cap X$.

A polynomial P is *compatible* with a cell $\mathcal{C}$ if $P|_{\mathcal{C}} \equiv 0$ or $P|_{\mathcal{C}}$ is non-vanishing. Equivalently P has a constant sign on $\mathcal{C}$.

Theorem

$P_1, \ldots, P_k$ polynomials $\implies \mathbb{R}^\ell = \bigcup_\alpha \mathcal{C}_\alpha$ – a finite union of cell $\mathcal{C}_\alpha$ pairwise compatible with P_j's.

Second theorem implies the first.

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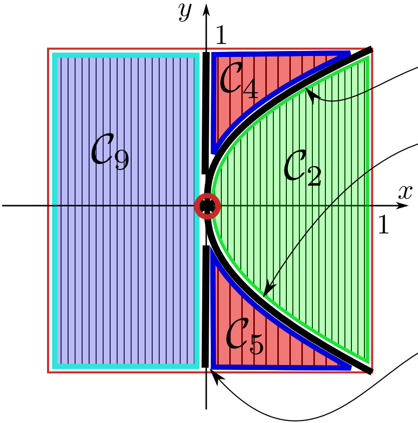
Cell Decompositions

$$\{y^2 = x\} \subset \{|x|, |y| < 1\}$$



$\mathcal{C}_1 = \{(x,y)|0 < x < 1, y = \sqrt{x}\}$

$\mathcal{C}_2 = \{0 < x < 1, -\sqrt{x} < y < \sqrt{x}\}$

$\mathcal{C}_3 = \{(x,y)|0 < x < 1, y = -\sqrt{x}\}$

$\mathcal{C}_4 = \{0 < x < 1, \sqrt{x} < y < 1\}$

$\mathcal{C}_5 = \{0 < x < 1, -1 < y < -\sqrt{x}\}$

$\mathcal{C}_6 = \{(0,0)\}$

$\mathcal{C}_7 = \{x = 0, 0 < y < 1\}$

$\mathcal{C}_8 = \{x = 0, -1 < y < 0\}$

$\mathcal{C}_9 = \{-1 < x < 0, -1 < y < 1\}$

The Yomdin-Gromov theorem

Theorem (Yomdin-Gromov Algebraic Lemma, 1987)

Let $X \subset [0,1]^\ell$ be a set of dimension μ defined by polynomial equations or inequalities of total degree β. Then for every $r \in \mathbb{N}$ there exists a collection of C^r-smooth maps $\phi_j : (0,1)^\mu \to X$ whose images cover X and $\|\phi_j\|_r \leqslant 1$. Moreover the number of maps is bounded by a constant $C = C(\ell, \mu, \beta, r)$.

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- The Y-G theorem is the key step in Yomdin's proof of Shub's entropy conjecture for smooth maps. It also plays a crucial role in Pila-Wilkie's work on the density of rational points in definable sets.
- Y-G is useful because it allows to do "Taylor approximations" on semialgebraic (or subanalytic) sets.
- The dependence of $C(\ell, \mu, \beta, r)$ on β and r is important for both ergodic and diophantine applications.

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A *cellular r-parametrization* of a definable set $X \subset \mathbb{R}^\ell$ is a collection $\Phi = \{\phi_\alpha : I_\alpha \to X\}$ of definable cellular C^r-smooth maps ϕ_α with $\|\phi_\alpha\|_r \leqslant 1$ such that $X = \cup_\alpha \phi_\alpha(I_\alpha)$.
Here $I_\alpha = I_{\alpha,1} \cdot \ldots \cdot I_{\alpha,\ell}$ (ℓ factors), where $I_{\alpha,j}$ are either $(0,1)$ or 0.

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Yomdin-Gromov Lemma, Binyamini-N. based on Pila-Wilkie pf. 2020

Every definable set $X \subset [0,1]^\ell$ admits a cellular r-parametrization.

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Every definable set $X \subset [0,1]^\ell$ admits a cellular r-parametrization.

Automatically implies uniformness in parameters.

Yomdin-Gromov complexification we would like to have

Denote $D(r) = \{|z| < r\}$. Let $0 < \delta < 1$.

C^r-smooth bounds on maps should be upgraded to δ-extendability

We say that a holomorphic map $f : D^\mu(1) \mapsto \mathbb{C}^\ell$ is δ-extendable if f can be holomorphically extended to $D^\mu(\delta^{-1})$.

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Cauchy formulas then imply estimates on C^r-norm of f on $D^\mu(1)$.

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Wanted result

Let $X \subset \mathbb{C}^\ell$ be a complex algebraic set of dimension μ and complexity β. Then there is a finite set of maps $\phi_j : D^\mu(1) \to X$ covering $X \cap D(1)^n$ s.t.

- ϕ_j are $1/2$-extendable with $\|\phi_j\|_{D^\mu(2)} \leqslant 2$, and
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Key (counter)example

For $X = \{xy = \epsilon\} \subset \mathbb{C}^2$ one needs $\sim \log\log \epsilon^{-1}$ such maps as $\epsilon \to 0$.

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Lemma (Schwartz-Pick)

Let $f : U \to U'$ be a holomorphic map between hyperbolic domains. Then

$$\operatorname{dist}(f(z), f(w); U') \leqslant \operatorname{dist}(z, w; U), \qquad \forall z, w \in U \tag{1}$$

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I.e. holomorphic maps between hyperbolic domains are contractions.

Back to our example

We want to cover $K = \{xy = \varepsilon, |x|, |y| \leqslant 1\}$ by $\phi_i(D(1))$, with $\phi_i : D(2) \to X = \{xy = \varepsilon, |x|, |y| < 2\}$.

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By projection $\pi : \mathbb{C}^2 \to \mathbb{C}$ to x-axis,

$$K \simeq \pi(K) = A(\varepsilon, 1) \subset A(\varepsilon/2, 2) = \pi(X) \simeq X,$$

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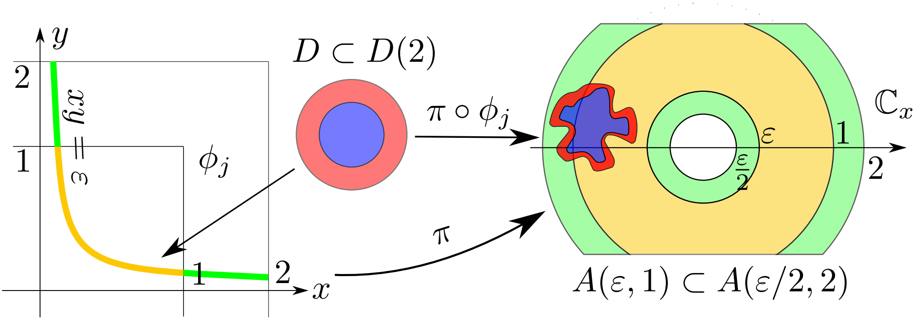

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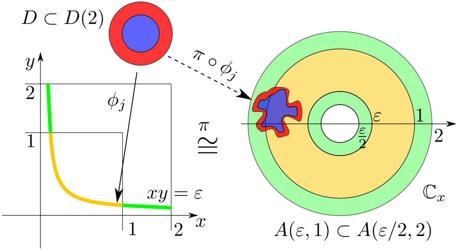

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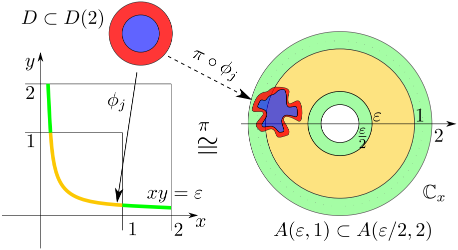

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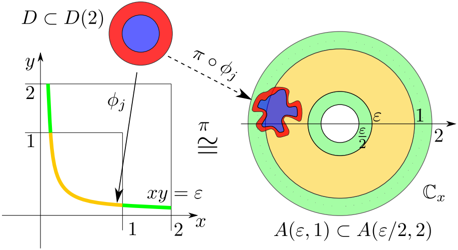

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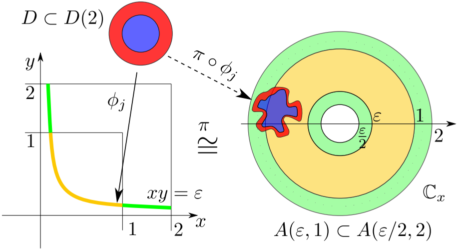

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Conclusion: to cover K by $\phi_j(D(1))$ we will need at least $\sim \log \log \varepsilon^{-1}$ maps!

However, one can cover K trivially by annulus $A(\epsilon, 1)$

Complex cells: Definition

- A complex cell $\mathcal{F} \subset \mathbb{C}$ of length 1 is one of

$$* = \{0\} \qquad D(r) = \{|z| < |r|\}$$

$$D_\circ(r) = \{0 < |z| < |r|\} \qquad A(r_1, r_2) = \{|r_1| < |z| < |r_2|\},$$

where $r \in \mathbb{C}$, $|r| > 0$ or, respectively, $r_1, r_2 \in \mathbb{C}$, $0 < |r_1| < |r_2|$.

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- A complex cell $\mathcal{F} \subset \mathbb{C}$ of length 1 is one of
$$* = \{0\} \qquad D(r) = \{|z| < |r|\}$$
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$$\mathcal{C} = \mathcal{C}_{1..\ell} \odot \mathcal{F} := \{\mathbf{x}_{1..\ell+1} : \mathbf{x}_{1..\ell} \in \mathcal{C}_{1..\ell},\ \mathbf{x}_{\ell+1} \in \mathcal{F}(\mathbf{x}_{1..\ell})\}$$
where $\mathcal{C}_{1..\ell}$ is a cell of length ℓ and the fibers $\mathcal{F}(\mathbf{x}_{1..\ell})$ are cells of length 1 with $r = r(\mathbf{x}_{1..\ell})$ (or $r_1 = r_1(\mathbf{x}_{1..\ell}), r_2 = r_2(\mathbf{x}_{1..\ell})$) being *holomorphic* bounded functions on $\mathcal{C}_{1..\ell}$ satisfying $r \neq 0$ (or $0 < |r_1| < |r_2|$, respectively).

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Example

$$D_\circ(1) \odot A(\mathbf{z}_1, 2) = \{\mathbf{z}_{1,2} : 0 < |\mathbf{z}_1| < 1,\ |\mathbf{z}_1| < |\mathbf{z}_2| < 2\}.$$

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Definition

A holomorphic function $F \in \mathcal{O}(\mathcal{C})$ is compatible with $\mathcal{C}$ if F is identically zero or non-vanishing on $\mathcal{C}$.

δ-extensions of complex cells

Holomorphicity means we can talk about analytic continuation.

δ-extensions of complex cells

For $0 < \delta < 1$ the δ-extension of a cell $\mathcal{C} = \mathcal{C}_{1..\ell} \odot \mathcal{F}$ is defined inductively by

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This is the principal new ingredient missing in the real context. The hyperbolic geometry of the pair $\mathcal{C} \subset \mathcal{C}^\delta$ plays the key role in our approach.

Complex cellular decomposition

If $f : \mathcal{C} \to \hat{\mathcal{C}}$ maps $\mathbf{z} \to \mathbf{w}$ we say that f is *prepared* if f is holomorphic and bounded on $\mathcal{C}$ and for $j = 1, \ldots, \ell$

$$\mathbf{w}_j = \mathbf{z}_j^{\mu_j} + \phi_j(\mathbf{z}_{1..j-1}), \qquad \mu_j \in \mathbb{N}_{>0}.$$

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Theorem (CPT=Cellular Parametrization Theorem)

Let $\mathcal{C}$ admits δ-extension and $F_1, \dots, F_k \in \mathcal{O}_b(\mathcal{C}^\delta)$. Then there exists a finite collection of prepared cellular maps $f_j : \mathcal{C}_j \to \mathcal{C}^\delta$ which admit δ-extensions such that the $f_j(\mathcal{C}_j^\delta) \subset \mathcal{C}^\delta$ are compatible with each F_i and $\mathcal{C} \subset \cup f_j(\mathcal{C}_j)$.

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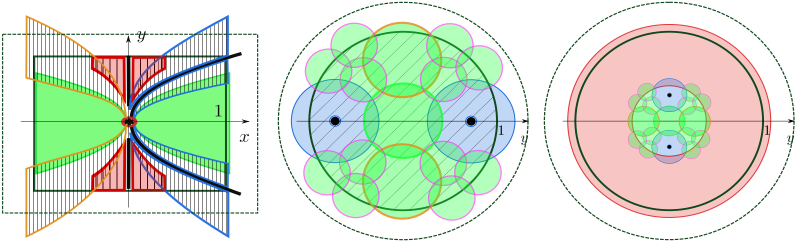
If $\mathcal{C}$, F_i are algebraic of complexity β, then the number of maps is $\operatorname{poly}_\ell(\beta, k, \delta)$ *and f_j, $\mathcal{C}_j$ are algebraic of complexity* $\operatorname{poly}_\ell(\beta, k)$.

CPT

$\{y^2 = x\} \subset \{|x|, |y| < 1\}$ $\{x = 1/2\}$ $\{x = -0.1\}$



1) $* \odot *$ 2) $* \odot D_\circ(1)$ 3) $D_\circ(1) \odot *$ 4) $D_\circ(1) \odot D(z/2)$

4-5) $D_\circ(1) \odot D_\circ(z/2)$ 7-12) $D_\circ(1) \odot D_\circ(z/4)$ 13) $D_\circ(0.4) \odot A(\frac{5}{4}z, 1)$

$\phi_1(*, *) = (0, 0)$ $\phi_2(z, *) = (z^2, z)$ $\phi_3(*, w) = (0, w)$

$\phi_4(z, w) = (z^2, w + z)$ $\phi_5(z, w) = (z^2, w)$ $\phi_6(z, w) = (z^2, w + iz)$

$\phi_j(z, w) = (z^2, w + c_j z), \quad j = 7, .., 12$ $\phi_{13}(z, w) = (z^2, w)$

Real Applications of CPT

If $\mathcal{C}$, $F_1, \ldots, F_k$ are real then $\mathbb{C}$C.D. implies real C.D.

Effective Yomdin-Gromov constant

The constant $C = C(\ell, \mu, \beta, r) = \mathrm{poly}_\ell(\beta) \cdot r^\mu$. Moreover, the maps ϕ_j can be chosen to be semialgebraic of complexity $\mathrm{poly}_\ell(\beta, r)$.

This implies tight bounds on the tail entropy and volume growth for analytic maps, thus solving an old Yomdin's conjecture (1991).
Applications to counting rational points on algebraic and transcendental varieties – Gal.

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Motivation 3: resolution of singularities

Theorem (Uniformization theorem)

Let $F_1, .., F_k \in \mathcal{O}_b(B)$. Then B can be covered by images of maps $f_j : B_j \to B$ such that $f_j^* F_i$ is a monomial times a unit. Moreover the maps are of a special form.

The following is a complex cells analogue

Monomialization Lemma

Let $F : \mathcal{C}^\delta \to \mathbb{C} \setminus \{0\}$ be holomorphic and bounded. Then on $\mathcal{C}$ we have $F = \mathbf{z}^\alpha \cdot U(\mathbf{z})$ where $\alpha \in \mathbb{Z}^\ell$ and U is holomorphic and

$$\operatorname{diam}(\operatorname{Re}\log U(\mathcal{C}); \mathbb{R}) < O_f(1) \cdot \rho, \qquad \operatorname{diam}(\operatorname{Im}\log U(\mathcal{C}); \mathbb{R}) < O_f(1),$$

with $|\alpha(F)|, O_F(1) = \operatorname{poly}_\ell(\beta)$ in algebraic case.

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An important Lemma from ROS

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Suppose F, G and $F - G$ are equal to a monomial times a unit. Then either F divides G or G divides F.

The B-M lemma plays a key role in the inductive proofs of ROS. It is qualitative: either local or compact settings.

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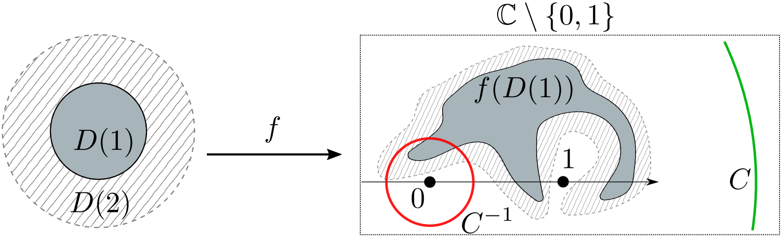

$$\text{Schwarz-Pick: } \operatorname{diam}\left(f\left(D(1)\right), \mathbb{C} \setminus \{0,1\}\right) \leq \operatorname{diam}\left(D(1), D(2)\right) = \log\sqrt{3}$$

Proof: Take $C > 0$ s.t.

$$\operatorname{dist}\left(\{|z| = C\}, \{|z| = C^{-1}\}; \mathbb{C} \setminus \{0,1\}\right) > \log\sqrt{3}.$$

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A quantitative result corresponding to the uniqueness result.

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Corollary: Big Picard Theorem

Let $f : D_\circ(1) \mapsto \mathbb{C} \setminus \{0, 1\}$ be a holomorphic function. Then f is meromorphic in $D_\circ$.

Proof: Either f or f^{-1} is bounded on $D_\circ(1/2)$.

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- If $F, G, F - G$ are compatible with $\mathcal{C}^\delta$, i.e. non-vanishing there, then $R := F/G$ satisfies the conditions of the lemma. The lemma then says that either F/G or G/F are bounded (compare with B-M) and gives bounds on R.

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- The lemma says roughly that a key building block of ROS "does not feel" the non-compactness of complex cells.

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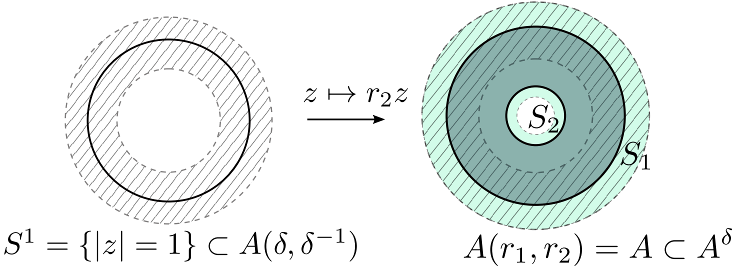
- The diameter of $A \subset A^\delta$ is unbounded as $r_1/r_2 \to 0$ (this was the whole point of allowing annuli). However the diameters of $S_1 = \{|z| = r_1\}$ and $S_2 = \{|z| = r_2\}$ in A^δ *are* bounded by $\rho = \rho(\delta) = \operatorname{diam}\left(S^1; A(\delta, \delta^{-1})\right)$.



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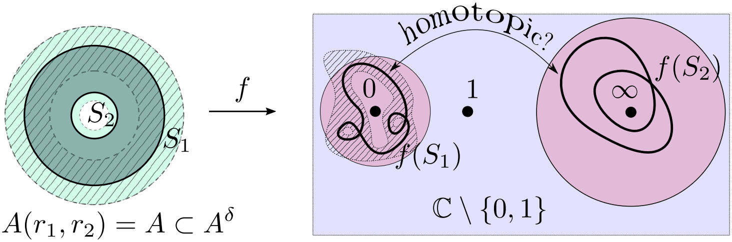

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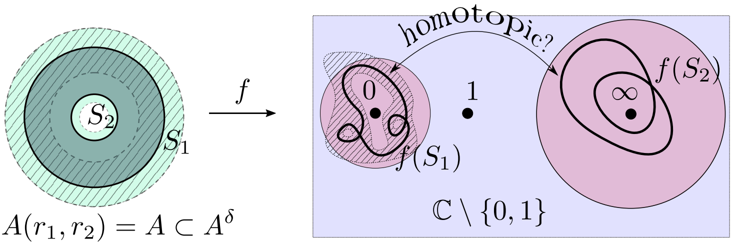
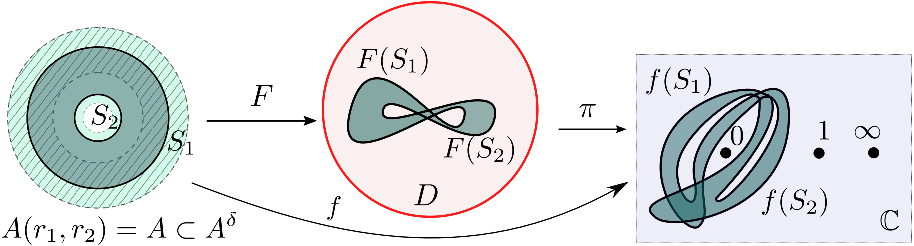
- S_1, S_2 are homotopic in A, so $f(S_1), f(S_2)$ are homotopic in $\mathbb{C} \setminus \{0,1\}$. But $f(S_1)$ is near 0 and $f(S_2)$ is near ∞, so they are in fact contractible.



- Therefore we may lift f to the universal cover, $F : \mathcal{C}^\delta \to D$ such that $\pi \circ F = f$. We claim that $\operatorname{diam}(F(\mathcal{C}); D) \leqslant 2\rho$.



Proof in the one-dimensional case (cont.)

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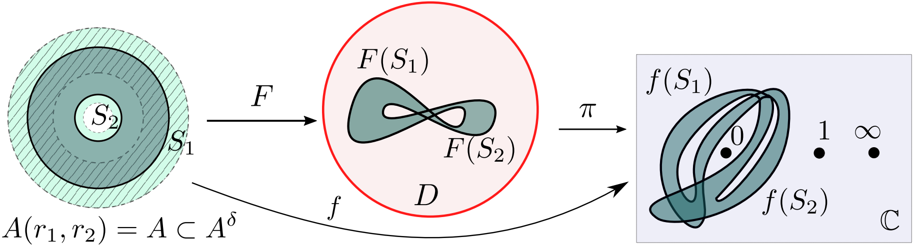

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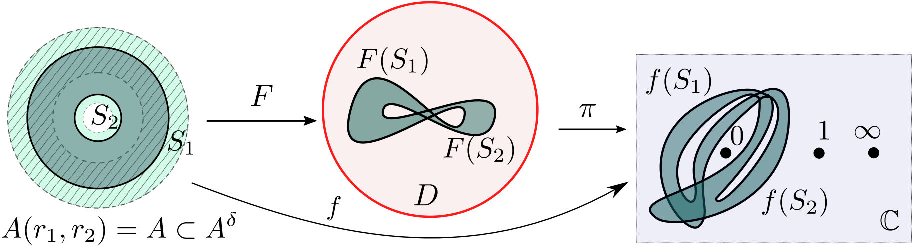

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- By Schwarz-Pick, $\operatorname{diam}(f(\mathcal{C}); \mathbb{C} \setminus \{0,1\} \leqslant \operatorname{diam}(F(\mathcal{C}); D) \leqslant 2\rho$. So it cannot be too close to both 0 and ∞ and we're done.