Finitely Presented Exponential Fields

Jonathan Kirby

University of East Anglia

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Abstract

In this talk I will outline the main ideas of the algebraic theory of exponential fields, in particular finitely presented extensions of exponential fields, and give some applications. On the number-theoretic side, I will explain a precise theorem which captures the folklore knowledge that Schanuel's conjecture answers all transcendence questions about exponentials and logarithms. On the model-theoretic side, I will give a simpler construction of Zilber's field, $\mathbb{B}$. Zilber used infinitary methods throughout his construction, but they are needed only at one point. As a corollary I answer a question of Macintyre and show that $\mathbb{B}$ is not model-complete. The proof shows that any algebraic theory of exponential varieties must be essentially more complicated than the situation of algebraic varieties. These complications do not arise in the real case (by model completeness) but it is an open question as to whether they arise in the complex case.
Most of the material in the talk comes from [Kir10a].

Outline

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Exponential fields

Definition

An *exponential field* (or *E-field*) is a field F of characteristic zero equipped with a homomorphism $\exp_F$ (also written exp, or $x \mapsto e^x$) from the additive group $\mathbb{G}_{\mathrm{a}}(F) = \langle F; + \rangle$ to the multiplicative group $\mathbb{G}_{\mathrm{m}}(F) = \langle F^{\times}; \cdot \rangle$.

Examples

- $\mathbb{R}_{\exp}$ and $\mathbb{C}_{\exp}$, exponential map given by the familiar power series
- Direct algebraic constructions

$\mathbb{R}_{\exp}$ is now well-understood:

- o-minimal and model complete (Wilkie) [Wil96]
- decidable if Schanuel's conjecture is true (Wilkie / Macintyre) [MW96]

The complex exponential field

Pure fields

- As a pure field, $\mathbb{C}$ is strongly minimal – simplest possible model-theoretic behaviour
- Good understanding of models (categoricity) and also of definable sets (varieties / constructible sets)
- $\mathbb{R}$ is unstable, so no control over models . . .
- . . . but o-minimal, so good structure theory for definable sets (semialgebraic sets)

Arithmetic

- $\mathbb{C}_{\exp}$ inteprets arithmetic: $\mathbb{Z} = \{y : \forall x[\exp(x) = 1 \to \exp(yx) = 1]\}$
- So $\mathbb{C}_{\exp}$ is undecidable
- Also unstable, no control over models
- No structure theory for all definable sets

Can the effects of arithmetic be contained? Is there *stable-like behaviour*?

Zilber's conjectures

Quasiminimality conjecture

- Zilber conjectured (1990s) that $\mathbb{C}_{\exp}$ is quasiminimal – every definable subset is either countable or co-countable.
- Would imply *generic stability* of the generic type (Pillay / Tanovic) [PT]
- Much work on the conjecture by Wilkie – still hard open problem.
- Alternative possibility that $\mathbb{R}$ is definable in $\mathbb{C}_{\exp}$ is also open.

Zilber's exponential field $\mathbb{B}$

- Zilber constructed a non-elementary theory Ψ of E-fields [Zil05]
- Ψ is uncountably categorical – very good understanding of models
- Conjecture: unique model $\mathbb{B}$ of cardinality $2^{\aleph_0}$ is isomorphic to $\mathbb{C}_{\exp}$
- Very hard conjecture – implies quasiminimality – not known to be false! – probably out of reach

My programme

- Categoricity of $\mathbb{B} \implies$ must be a good algebraic theory of E-fields
- Should be a good structure theory for some definable sets
- Study the algebra of exponential fields, in particular finitely generated extensions
- Classify these extensions in terms of finite presentations
- Develop corresponding theory of exponential algebraic varieties
- Use this to understand $\mathbb{B}$, and its first-order theory
- Much of the theory should apply unconditionally to $\mathbb{C}_{\exp}$

Outline

Constructing an exponential map

General construction

- F a field, characteristic 0.
- Choose a $\mathbb{Q}$-linear basis $\{b_i \mid i \in I\}$ of F
- For each $i \in I$, choose $c_{i,1} \in F$, and define $\exp(b_i) = c_{i,1}$
- $\exp(b_i/m)$ must be an m^{th} root of $c_{i,1}$
- For each $i \in I$ and $m \in \mathbb{N}$ choose $c_{i,m} \in F$ such that for any $r, m \in \mathbb{N}$, we have $c_{i,rm}^r = c_{i,m}$.
- Define $\exp(\frac{1}{m}\sum r_i b_i) = \prod c_{i,m}^{r_i}$. The coherence condition shows that exp is well-defined.

Extensions of E-fields

- $F \subseteq F(\alpha)$ extension of fields. Want to extend exp from F to $F(\alpha)$
- If instead we want to extend a derivation ∂, enough to specify one equation for $\partial\alpha$.
- Similarly for extensions of difference fields
- But $F(\alpha)/F$ is an infinite dimensional $\mathbb{Q}$-vector space, so lots of freedom for extending exp

Partial exponential fields

- There is a lot of freedom. Cannot hope to describe all possible exponential fields, or all extensions of exponential maps even to finitely generated extension fields.
- Instead we define exponential maps a little at a time, looking for good properties.

Definition

A *partial exponential field* is a field F of characteristic zero with $\exp_F : D(F) \to \mathbb{G}_m(F)$, where $D(F)$ is some $\mathbb{Q}$-linear subspace of F.

Examples

- Any E-field F is a partial E-field, with $D(F) = F$
- $\mathbb{Q}_0$ is $\mathbb{Q}$ with $D(\mathbb{Q}_0) = \{0\}$, and trivial exponential map
- SK is $\mathbb{Q}^{ab}(2\pi i)$, with $D(SK) = \mathbb{Q} \cdot 2\pi i$, and $\exp(2q\pi i) = e^{2q\pi i}$

Free constructions 1

Given a partial E-field, F, we want to extend it to a (total) E-field in as free a way as possible. We assume that $\exp_F$ already has a cyclic kernel, and we do not want to extend it. Say $SK \subseteq F$. So the image of $\exp_F$ contains all roots of unity.

1. Let $\{b_i \mid i \in I\} \subseteq F$ be representatives of a $\mathbb{Q}$-linear basis for $F/D(F)$.
2. Choose $\{c_{i,n} \mid i \in I, n \in \mathbb{N}\}$ in a field extension such that the $c_{i,1}$ are algebraically independent over F, and for each i, $(c_{i,n})_{n\in\mathbb{N}}$ is a coherent system of roots of $c_{i,1}$.
3. Each $r \in F$ is a finite sum of the form $r_0 + \frac{1}{n}\sum m_i b_i$ for some $r_0 \in D(F)$, $n \in \mathbb{N}$, and some $m_i \in \mathbb{Z}$, so define $\exp(r_0 + \frac{1}{n}\sum m_i b_i) = \exp_F(r_0)\prod c_{i,n}^{m_i}$.
4. Get new partial E-field F_1 with $D(F_1) = F$, generated as a field by $D(F_1) \cup \exp(D(F_1))$.
5. Iterate to get a total E-field at stage ω. Call it F^E .

It is easy to see that F^E is well-defined, and the exponential map is *as free as possible*. There are no unnecessary algebraic relations between exponentials.

Free constructions 2

With one small change we can get an algebraically closed E-field.

1. Let $\{b_i \mid i \in I\} \subseteq F$ be representatives of a $\mathbb{Q}$-linear basis for $F/D(F)$.
2. Choose $\{c_{i,n} \mid i \in I, n \in \mathbb{N}\}$ in a field extension such that the $c_{i,1}$ are algebraically independent over F, and for each i, $(c_{i,n})_{n\in\mathbb{N}}$ is a coherent system of roots of $c_{i,1}$.
3. Each $r \in F$ is a finite sum of the form $r_0 + \frac{1}{n}\sum m_i b_i$ for some $r_0 \in D(F)$, $n \in \mathbb{N}$, and some $m_i \in \mathbb{Z}$, so define $\exp(r_0 + \frac{1}{n}\sum m_i b_i) = \exp_F(r_0)\prod c_{i,n}^{m_i}$.
4. Get new partial E-field F_1 with $D(F_1) = F$, which as a field is the algebraic closure of $D(F_1) \cup \exp(D(F_1))$.
5. Iterate to get a total E-field at stage ω. Call it F^{EA}.

A similar construction gives an E-field F^{ELA} which is algebraically closed and has a surjective exponential map, and the same kernel as F.

Definition

An exponential field which is algebraically closed and has a surjective exponential map is called an ELA-field.

Free constructions 3

Given a partial E-field F, with cyclic kernel, we freely construct extensions

$$F \subseteq F^E \subseteq F^{EA} \subseteq F^{ELA}$$

Theorem

1. *F^E and F^{EA} are well-defined extensions of F.*
2. *If F is finitely generated as a partial E-field, or is a finitely generated extension of a countable ELA-field, then F^{ELA} is well-defined.*

Proof.

1 is easy. 2 uses the Thumbtack Lemma of Zilber and Bays. [Zil06], [BZ07] □

Strong extensions

Definition

Let F be a partial E-field, $\bar{x}$ a finite tuple from $D(F)$, and B a subset of F. Define the *relative predimension function*

$$\delta(\bar{x}/B) = \mathrm{td}(\bar{x}, \exp(\bar{x})/B, \exp(B)) - \mathrm{ldim}_{\mathbb{Q}}(\bar{x}/B)$$

Definition

An extension $F \subseteq F_1$ of partial E-fields is *strong*, written $F \lhd F_1$, iff for every tuple $\bar{x}$ from $D(F_1)$, we have $\delta(\bar{x}/F) \geqslant 0$.

Lemma

For any partial E-field F, $F \lhd F^E$ and $F \lhd F^{ELA}$.

Theorem (Main technical theorem)

If F is finitely generated as a partial E-field, or is a finitely generated extension of a countable ELA-field, and $F \lhd K$ where K is an ELA-field, then the ELA-subfield of K generated by F is isomorphic to F^{ELA}.

Schanuel's conjecture

Schanuel's conjecture

For any $a_1, \ldots, a_n \in \mathbb{C}$, $\mathrm{td}(a_1, \ldots, a_n, e^{a_1}, \ldots, e^{a_n}) - \mathrm{ldim}_{\mathbb{Q}}(a_1, \ldots, a_n) \geqslant 0$.

Equivalently, $\mathbb{Q}_0 \lhd \mathbb{C}_{\mathrm{exp}}$. In fact, *SK* embeds in $\mathbb{C}_{\mathrm{exp}}$, and one can show Schanuel's conjecture is equivalent to $SK \lhd \mathbb{C}_{\mathrm{exp}}$.

Definition (Ritt 1948? [Rit48], Chow 1999 [Cho99])

The ELA-field $\mathbb{L}$ of *Liouvillian numbers* is the smallest ELA-subfield of $\mathbb{C}_{\mathrm{exp}}$. Its smallest EL-subfield $\mathbb{E}$ consists of all complex numbers with a closed form representation in terms of exp and the principal logarithm Log.

Corollary

If Schanuel's conjecture is true, then $\mathbb{L} \cong SK^{ELA}$. Furthermore, there is an algorithm for deciding transcendence questions in $\mathbb{L}$, and in $\mathbb{E}$.

Outline

Finitely generated partial E-field extensions

Fix an ELA-field, F. Consider a partial E-field extension $F \subseteq F_1$ generated by $\bar{x} = (x_1 \ldots, x_n) \in D(F_1)$. Let $V = \mathrm{Loc}(\bar{x}, \exp^{\bar{x}} / F)$, the algebraic locus.

- We may assume $\bar{x}$ is $\mathbb{Q}$-linearly independent over F. We say V is *additively free*.
- We assume the extension does not extend the kernel. Equivalently, $\exp(\bar{x})$ is multiplicatively independent over F. Say V is *multiplicatively free*.

Lemma (Application of thumbtack lemma)

Replacing $\bar{x}$ by $\bar{x}/N$ for some $N \in \mathbb{N}$ if necessary, F_1 is determined up to isomorphism as an extension of F by V.

So every finitely generated *partial* E-field extension of F is determined by finite data: the subvariety V of $F^n \times (F^\times)^n$ (or a finite list of polynomials determining it).

Finitely presented ELA-extensions

- We have $F \subseteq F_1$, with F an ELA-field and F_1 a finitely generated partial E-field extension.
- $F \subseteq F_1^{ELA}$ is a finitely generated ELA-field extension, well-defined by the main technical theorem.
- Write $F_1^{ELA} = F|V$ where $V = \mathrm{Loc}(\bar{x}, \exp(\bar{x})/F)$ as before.

Definition

Suppose $F \subseteq K$ is a finitely generated extension of ELA-fields. If there is F_1 a finitely generated partial E-field extension of F such that $K \cong_F F_1^{ELA}$, we say that K is a finitely presented ELA-extension of F. We say that an appropriate variety V is the presentation.

Theorem

If $F \lhd K$ is a finitely generated kernel-preserving strong extension of (countable) ELA-fields, then it is finitely presented.

Warning: there are $2^{\aleph_0}$ finitely generated kernel-preserving strong E-field extensions of most countable E-fields!

Corollary

If Schanuel's conjecture is true then every finitely generated ELA-subfield of $\mathbb{C}_{\exp}$ is finitely presented. (We could say that $\mathbb{C}_{\exp}$ is locally finitely presented.)

- Given a (countable) ELA-field F, the finitely-generated strong extensions are of the form $F \lhd F|V$ for certain varieties V (called rotund).
- The extension $F \lhd F|V$ is exponentially algebraic if $\dim V = n$ (where $V \subseteq F^n \times (F^\times)^n$, and V is additively and multiplicatively free).
- Otherwise exponentially transcendental, for example $n = 1$, $V = F \times F^\times$.

Let $\mathbb{C}_0$ be the subfield of $\mathbb{C}_{\exp}$ consisting of all the exponentially algebraic numbers. Then $\mathbb{L} \subseteq \mathbb{C}_0$, but for example if $e^a = a$ then $a \in \mathbb{C}_0$ and (under Schanuel's conjecture) $a \notin \mathbb{L}$. In fact, $\mathbb{C}_0$ is countable.

Theorem

1. *Unconditionally, $\mathbb{C}_0 \lhd \mathbb{C}_{\exp}$.*
2. *Every finitely generated ELA-extension of $\mathbb{C}_0$ in $\mathbb{C}_{\exp}$ is finitely presented.*

Outline

Amalgamation

- Fix a (countable) ELA-field, F, for example SK^{ELA}.
- The finitely generated strong ELA-extensions are $F \lhd F|V$ for rotund V.
- There are only countably many such V, so only countably many such extensions.
- One can easily show $(F|V)|W \cong F|(V \times W) \cong (F|W)|V$, so we have amalgamation.
- This is enough to use the Fraissé / Hrushovski technique to get a universal domain for strong kernel-preserving extensions of F.
- If we restrict to those exponentially algebraic strong extensions which do not already occur infinitely many times within F, we get an ELA-field called the strong exponential algebraic closure of F, written $F^{\sim}$.

Zilber's exponential fields

Let $B_0 = SK^{\sim}$. For a cardinal κ, let F_κ be the field of rational functions in κ indeterminates over B_0, and let $B_\kappa = F_\kappa{}^{\sim}$.
These B_κ are Zilber's exponential fields, and $\mathbb{B} = B_{2^{\aleph_0}}$.

Axiomatic approach

Instead of this construction, Zilber gave a list of axioms and proved categoricity.

Axioms for Zilber's exponential fields

1. B is an ELA-field.
2. B has cyclic kernel.
3. The conclusion of Schanuel's conjecture holds (equivalently, $SK \triangleleft B$).
4. B is strongly exponentially algebraically close, which can be axiomatized as: for every free, rotund V of dimension n and every finite $\bar{a}$ there is $\bar{x}$ such that $(\bar{x}, \exp(\bar{x}))$ is generic in V over $\bar{a}$.
5. For each such V and $\bar{a}$, there are only countably many such $\bar{x}$.

The axioms can be expressed as an $L_{\omega_1,\omega}(Q)$-sentence.

In [Kir10b] I show how to minimize the usage of non-first order axioms, and also prove that Zilber's fields are the prime models over exponential transcendence bases of their common complete first-order theory.

Some model theory of $\mathbb{B}$

If $F_0 \subseteq \mathbb{B}$ is a strong ELA-subfield, or $F_0 = \emptyset$, then types of finite tuples over F_0 are given by boolean combinations of formulas of the form

$$(\exists \bar{y})(\bar{x}, \bar{y}, \exp(\bar{x}), \exp(\bar{y})) \in V$$

where V runs over rotund varieties defined over F_0. So for the types over such F_0 which are realised in $\mathbb{B}$, there is some quantifier elimination (near model completeness). This does not apply to types over other parameter sets, or types which are not realised in $\mathbb{B}$ (extending the kernel).

Example

The formula $x = \exp(x)$ defines a strongly minimal set in $\mathbb{B}$. Furthermore, it has the structure of a pure set – the only definable subsets are finite and cofinite, and every n-tuple has the same type over $\emptyset$.

More work should be done to understand the types further.

Outline

Model completeness

Definition

A first-order theory T is model complete iff any of the following equivalent condtions holds:

1. Every formula is equivalent to both a universal formula and an existential formula.
2. Every embedding of models is an elementary embedding.
3. The diagram of any model is a complete theory.

A structure is model complete iff its first-order theory is model complete.

- Wilkie showed $\mathbb{R}_{\exp}$ is model complete, and also $\mathbb{R}_{\exp,\sin\restriction_{[-\pi,\pi]}}$ (an o-minimal version of $\mathbb{C}_{\exp}$)
- $(\mathbb{Z}; +, \cdot)$ does not have QE to any finite number of quantifier alternations.
- Although $\mathbb{C}_{\exp}$ interprets arithmetic, it does not immediately follow that it is not model complete.
- Macintyre & Marker used the Baire category theorem to show $\mathbb{C}_{\exp}$ is not model complete. [Mar06]
- That topological argument does not apply to $\mathbb{B}$.

Arithmetic and Geometric model completeness

Macintyre and Marker's proof is connected to $\mathbb{Q}$, $\mathbb{Z}$, and arithmetic issues. It says nothing about exponential algebraic varieties and geometric questions.

Definition

Let M be a structure which interprets arithmetic, $(\mathbb{Z}; +, \cdot)$.

- M is arithmetically model-complete if every definable subset of $\mathbb{Z}$ in M is definable by both a universal formula and an existential formula.
- M is geometrically model-complete if whenever M_1, M_2 are elementarily equivalent to M and the realisations of $\mathbb{Z}$ in both are standard, and $M_1 \subseteq M_2$ then $M_1 \preccurlyeq M_2$.

I believe Macintyre and Marker's proof shows $\mathbb{C}_{\exp}$ is not arithmetically model complete, but it is open whether $\mathbb{C}_{\exp}$ is geometrically model complete.

Theorem

$\mathbb{B}$ *is not geometrically model complete.*

Proof of non-model completeness

We construct an embedding $B_1 \hookrightarrow B_1$ which is not strong, and hence not elementary.

1. Let $V \subseteq F^3 \times (F^\times)^3$ be the intersection of 3 generic hyperplanes, defined over SK^{ELA}, and let $(a_1, a_2, a_3, e^{a_1}, e^{a_2}, e^{a_3}) \in V(B_1)$.
2. Let K_1 be the smallest ELA-subfield of B_1 containing a_1.
3. We show $a_2, a_3 \notin K_1$.
4. So in K_1, a_1 is exponentially transcendental (but not in B_1).
5. Let t be exponentially transcendental in B_1, and let F_1 be the smallest ELA-subfield of B_1 containing t. Then there is an isomorphism $F_1 \cong K_1$ taking t to a_1.
6. Using an asymmetric amalgamation lemma (as in Hrushovski's constructions), we show $K := K_1^{\sim}$ can be embedded into B_1, over K. Now $K_1 \lhd K$, but $K_1 \ntriangleleft B_1$, so $K \ntriangleleft B_1$. In particular, $a_2, a_3 \notin K$.
7. But $K \cong B_1$.

I suspect $\mathbb{B}$ is also not arithmetically model complete.

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