Decidability via the tilting correspondence

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Motivation: Function Field Analogy

Characteristic 0		Positive characteristic
$\mathbb{Q}$	vs	$\mathbb{F}_p(t)$
$\mathbb{Q}_p$	vs	$\mathbb{F}_p((t))$

$p \to \infty$

(1) Ax-Kochen/Ershov.
$\prod \mathbb{Q}_p/U \equiv \prod \mathbb{F}_p((t))/U$

(2) Artin's Conjecture.

$e \to \infty$

(1) $\mathbb{Q}_p(p^{1/n})$ vs $\mathbb{F}_p((t))$
(Krasner-Kazhdan-Deligne)

(2) $\mathbb{Q}_p(p^{1/p^\infty})$ vs $\mathbb{F}_p((t))^{1/p^\infty}$.
(Fontaine-Wintenberger)

(3) Tilt (Scholze)

(4) Model theory?

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Infinitely ramified extensions

Questions:
(a) Is $\mathrm{Th}_{\exists}\mathbb{Q}_p(p^{1/p^\infty})$ decidable in L_{val}?
(b) Is $\mathrm{Th}_{\exists}\mathbb{Q}_p(\zeta_{p^\infty})$ decidable in L_{val}? (Macintyre '86)
(c) Is $\mathrm{Th}_{\exists}\mathbb{Q}_p^{ab}$ decidable in L_{val}? (Koenigsmann ICM '18)
What about the full theories?

Program: Relate these questions with their positive characteristic analogues.

(a) Is $\mathrm{Th}_{\exists}\mathbb{F}_p((t))^{1/p^\infty}$ decidable in L_t? (Kuhlmann-Rzepka '21)
(b) Is $\mathrm{Th}_{\exists}\overline{\mathbb{F}}_p((t))^{1/p^\infty}$ decidable in L_t?
What about the full theories?

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Perfectoid fields

Definition

(K, v) with $\text{res}(k) = p$ is perfectoid if:

1. It is complete and $\text{rank}(\Gamma_v) = 1$.
2. Γ_v is non-discrete.
3. $\text{Frob} : \mathcal{O}_K/p \to \mathcal{O}_K/p : x \mapsto x^p$ is surjective.

Example

1. p-adic completions of $\mathbb{Q}_p(p^{1/p^\infty})$ and $\mathbb{Q}_p(\zeta_{p^\infty})$.
2. t-adic completion of $\mathbb{F}_p((t))^{1/p^\infty}$.

Tilting

K perfectoid field.

Construction:

1. Define $K^\flat = \varprojlim_{x \mapsto x^p} K$ (multiplicative monoid). Elements look like $(x, x^{1/p}, x^{1/p^2}, \ldots)$.
2. $(x_n)_{n\in\mathbb{N}} + (y_n)_{n\in\mathbb{N}} = (\lim_{m\to\infty}(x_{n+m} + y_{n+m})^{p^m}_{n\in\mathbb{N}})$.
3. $K^\flat$ is the tilt of K ; $v^\flat((x, x^{1/p}, \ldots)) := vx$.

Example

$\widehat{\mathbb{Q}_p(p^{1/p^\infty})}^\flat = \widehat{\mathbb{F}_p((t))^{1/p^\infty}} = \widehat{\mathbb{Q}_p(\zeta_{p^\infty})}^\flat$.

Remarks:

(1) $\Gamma_v = \Gamma_{v^\flat}$ and $k_v = k_{v^\flat}$.

(2) $\mathcal{O}_K/p \cong \mathcal{O}_{K^\flat}/\pi$, for some $\pi \in \mathcal{O}_{K^\flat}$.

Untilting

Setting

F: char.p, perfectoid (e.g. $F = \widehat{\mathbb{F}_p((t))^{1/p^\infty}}$).

Goal: Describe the untilts K of F (viz. $K^\flat \overset{\iota}{\cong} F$) "intrinsically".

Build $W(\mathcal{O}_F)$="Witt vectors over $\mathcal{O}_F$".
Distinguished element:
An element $\xi \in W(\mathcal{O}_F)$ such that $\xi = (\xi_0, \xi_1, \dots) \in \mathcal{O}_F^\omega$ with $\xi_0 \in \mathfrak{m}$ and $\xi_1 \in \mathcal{O}_F^\times$.

Theorem (Fargues-Fontaine)

Distinguished elements/units $\leftrightarrow$ **Untilts** $/ \cong$

$(\rightarrow)$: $W(\mathcal{O}_F)/(\xi) = \mathcal{O}_K$.
$(\leftarrow)$: Say $K^\flat = F$; take $\theta : W(\mathcal{O}_F) \to \mathcal{O}_K$ and $\text{Ker}(\theta) = (\xi)$.
cf. Rideau-Scanlon (continuous logic)

Space Y_F of untilts: Some model-theoretic properties

Notation: F char. p perfectoid.
$\overline{Y_F}$:="set of untilts of F / $\cong$".
The points of $\overline{Y_F}$ are $x = (K_x, \iota_x)$ with $K_x^\flat \overset{\iota}{\cong} F$.

Fargues-Fontaine metric: $d(x, y) := |\xi_y(x)|_x$.
Fact: $(\overline{Y_F}, d)$ is a complete ultrametric space. (Fargues-Fontaine)
Observation: $x_n \overset{d}{\to} x \neq 0 \Rightarrow K_{x_n} \to K_x$.

$Z_F := Y_F / \equiv$.
Fact: Usually, $|Z_F| = \mathfrak{c}$ (e.g., when $F = \widehat{\mathbb{F}_p((t))^{1/p^\infty}}$).
Moral: We cannot expect K to be decidable relative to $K^\flat$.

Computable untilts

Setting
F: char.p, perfectoid; $R_0 \subseteq \mathcal{O}_F$ computable.
(e.g. $F = \widehat{\mathbb{F}_p((t))^{1/p^\infty}}$ and $R_0 = \mathbb{F}_p[t]$.)

Distinguished elements/units $\leftrightarrow$ **Untilts** $/ \cong$ (Fargues-Fontaine)

Definition

K perfectoid such that $K^\flat = F$. We say that K is an R_0-computable untilt of F if there exists ξ such that $\mathcal{O}_K = W(\mathcal{O}_F)/\xi$, where $\xi = (\xi_0, \xi_1, \ldots)$ is R_0-*computable*, i.e.

1. $\xi_n \in R_0$ for all $n \in \mathbb{N}$.
2. $\mathbb{N} \to R_0 : n \mapsto \xi_n$ is recursive.

Cf. Computable reals/p-adic integers.

Relative decidability for perfectoid fields

Theorem (van den Dries 80's, unpublished)

(K, v) mixed characteristic henselian. Then $Th(K, v)$ in L_{val} depends on $Th(\mathcal{O}_K/p^n)$ $(n \in \mathbb{N})$ in L_{rings} & $Th(\Gamma_v, vp)$ in $L_{oag} \cup \{vp\}$.

Remarks:

(1) K is **not** decidable relative to $\mathcal{O}_K/p^n$ $(n \in \mathbb{N})$ and (Γ_v, vp).

(2) If $(\mathcal{O}_K/p^n)_{n\in\omega}$ is *uniformly decidable* and (Γ_v, vp) is decidable, then so is K.

Theorem (Transfer for perfectoids)

K is an R_0-computable untilt of $K^\flat$ (where $R_0 \subseteq \mathcal{O}_{K^\flat}$ computable). If $K^\flat$ is decidable in $L_{val}(R_0)$, then K is decidable in L_{val}.

Emphasis: Parameters.

Relative decidability for perfectoid fields (proof)

Theorem (Transfer for perfectoids)

K is an R_0-computable untilt of $K^\flat$ (where $R_0 \subseteq \mathcal{O}_{K^\flat}$ computable). If $K^\flat$ is decidable in $L_{val}(R_0)$, then K is decidable in L_{val}.

Proof Sketch.

Key idea 1: $\mathcal{O}_K/p^n \cong W_n(\mathcal{O}_{K^\flat})/(\xi \mod (p^n))$ (Fargues-Fontaine) where $\xi \mod p^n = (\xi_0, ..., \xi_{n-1})$ and $\xi_i \in R_0$.
Key idea 2: Use van den Dries.
Subtlety 3: $(\mathcal{O}_K/p^n)_{n\in\mathbb{N}}$ is **uniformly** decidable.
Get a *uniformly recursive* sequence of interpretations $(\Gamma_n)_{n\in\omega}$ of $\mathcal{O}_K/p^n$ in $K^\flat$. (computable $\xi = (\xi_0, \xi_1, ...)$)

□

Remark: (Parameters) Find Γ: p-divisible and decidable with $(\Gamma, 1)$ undecidable. Form $F = \mathbb{F}_p((t^\Gamma))$, which is decidable in L_{val}. Take $K = \mathrm{Frac}(W(\mathcal{O}_F)/([t] - p))$; (Γ_v, vp) is undecidable.

Corollary

(a) *Assume* $\mathbb{F}_p((t))^{1/p^\infty}$ *is (existentially) decidable in* L_t.
Then $\mathbb{Q}_p(p^{1/p^\infty})$ *and* $\mathbb{Q}_p(\zeta_{p^\infty})$ *are (existentially) decidable in* L_{val}.
(b) *Assume* $\overline{\mathbb{F}}_p((t))^{1/p^\infty}$ *is (existentially) decidable in* L_t.
Then $\mathbb{Q}_p^{ab}$ *is (existentially) decidable in* L_{val}.

Proof Sketch.

We can check that these are $\mathbb{F}_p[t]$-computable untilts.
$\mathbb{Q}_p(p^{1/p^\infty})$: $\xi = [t] - p$.
$\mathbb{Q}_p(\zeta_{p^\infty}), \mathbb{Q}_p^{ab}$: $\xi = [(t+1)]^{p-1} + ... + [t+1] + 1$.
Computable Witt vectors. □

Existential problem over $\mathbb{F}_p((t))^{1/p^\infty}$:
$X \to \text{Spec}(\mathbb{F}_p[t])$ of finite type; $X^{(p^n)} = n - th$ Frobenius twist.
Problem:
Is there n such that $X^{(p^n)}(\mathbb{F}_p((t))) \neq \emptyset$?
Remarks:
For each fixed n, we have an algorithm relying on resolution (Denef-Schoutens). However, we need *a priori* knowledge of n.

Aside: Tilting equivalence

Setting: K perfectoid field with tilt $K^\flat$.
$K - perf = \{$category of perfectoid field extensions of $K\}$.
$K^\flat - perf = \{$category of perfectoid field extensions of $K^\flat\}$.

Theorem (Tilting equivalence)

$K - perf \simeq K^\flat - perf.$

Description of the functors:
$(\rightarrow) : L \mapsto L^\flat$.
$(\leftarrow) : M \mapsto W(\mathcal{O}_M) \otimes_{W(\mathcal{O}_{K^\flat})} K$.
Remarks:
(1) If $L = M^\sharp$, then $\xi_L = \xi_K$.
(2) If K is a computable untilt of $K^\flat$, then L is a computable untilt of $L^\flat$.
(3) Respects ramification, inertia and field extension degrees.
(4) Extends to algebras; respects finite extensions ($G_K \cong G_{K^\flat}$).

Applications: Tame fields of mixed characteristic

Fact: Set $\Gamma = \frac{1}{p^\infty}\mathbb{Z}$. Then $\mathbb{F}_p(\!(t^\Gamma)\!)$ is decidable in L_t.
(Lisinski '21 building on Kuhlmann '16 and Kedlaya '06)

Corollary

The field $\mathbb{Q}_p(p^{1/p^\infty})$ admits a decidable maximal immediate extension.

Proof Sketch.

Set $K = \mathbb{F}_p(\!(t^\Gamma)\!)^\sharp$. We have $\xi_K = [t] - p$ (computable).

$$\mathbb{F}_p(\!(t^\Gamma)\!)^\sharp \qquad\qquad \mathbb{F}_p(\!(t^\Gamma)\!)$$

$$\widehat{\mathbb{Q}_p(p^{1/p^\infty})} \qquad\qquad \widehat{\mathbb{F}_p(\!(t)\!)^{1/p^\infty}}$$

Now use the transfer theorem for perfectoids. □

Remark:
Similar result for $\mathbb{Q}_p(\zeta_{p^\infty})$ (for $\mathbb{Q}_p^{ab}$ it was already known).

Applications: Congruences modulo p

Fact: $\mathbb{F}_p((t))^{1/p^\infty}$ is $\exists$-decidable in L_{val}. (Anscombe-Fehm '16)

Theorem

$Th_\exists(\mathbb{Z}_p[p^{1/p^\infty}]/p)$ is decidable.

Proof Sketch.

Key idea 1: $\mathbb{Z}_p[p^{1/p^\infty}]/p \cong \mathbb{F}_p[t^{1/p^\infty}]/t$. $(\mathcal{O}_K/p = \mathcal{O}_{K^\flat}/t)$

Key idea 2:

$$\mathbb{Z}_p[p^{1/p^\infty}]/p \models \exists x \; f(x) = 0 \wedge g(x) \neq 0$$

$$\iff \mathbb{F}_p[[t]]^{1/p^\infty} \models \exists x \; vg(x) < vf(x)$$

For $\Leftarrow$ move $I := (vg(x), vf(x))$ via an embedding so that $1 \in I$. Then reduce modulo t.

Key idea 3: Use Anscombe-Fehm. □

Remark: Similar results for $\mathbb{Q}_p(\zeta_{p^\infty})$ and $\mathbb{Q}_p^{ab}$ (use Kronecker-Weber).

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The theory of *all* p-adic fields (fixed p)

Open problem: Understand $\mathrm{Th}(\{K : [K : \mathbb{Q}_p] < \infty\})$ (fixed p).
First step: Understand the asymptotic theory/ultraproducts.

Theorem

The asymptotic theory of $\{K : [K : \mathbb{Q}_p] < \infty\}$ *is undecidable in* L_{val} *with cross-section.*

cf.

(1) Each indvidual K is decidable. (Ax-Kochen/Ershov '65)
(2) The (asymptotic) theory of $\{\mathbb{Q}_p : p \in \mathbb{P}\}$ is decidable with cross-section (Ax '67).
(3) The (asymptotic) theory of *all unramified* extensions of $\mathbb{Q}_p$ with cross-section. (Ax '67)
(4) $\mathbb{F}_p((t))$ is undecidable with cross-section. (Becker-Denef-Lipschitz '79)
(5) $\mathbb{F}_p((t))$ is $\exists$-undecidable with cross-section. (Pheidas '87)

Proof Sketch

Theorem

Fix $p \in \mathbb{P}$. The asymptotic theory of $\{K : [K : \mathbb{Q}_p] < \infty\}$ is undecidable in L_{val} with cross-section.

Proof Sketch.

1. Encode the asymptotic theory of $\{K : [K : \mathbb{Q}_p] = e(K/\mathbb{Q}_p) < \infty\}$.
2. For $K/\mathbb{Q}_p$ tot. ramified, we get $\mathcal{O}_K/p = \mathbb{F}_p[t]/t^e$.
3. Encode the asymptotic theory of $\{\mathbb{F}_p[t]/t^n : n \in \mathbb{N}\}$, with a predicate for powers of t.
4. Encode the asymptotic theory of $\{([n] \cup \infty; 1, \infty, +, |_p) : n \in \mathbb{N}\}$.
5. Use that the latter is undecidable by encoding $\mathrm{Th}_{\exists^+}(\mathbb{N}; 0, 1, +, |_p)$.

$\square$

cf. Pheidas' undecidability proof for $\mathbb{F}_p((t))$ with cross-section.

Pheidas' proof

Key idea: Interpret $(\mathbb{N}, +, |_p)$ in $\mathbb{F}_p((t))$ with cross-section.

Lemma (Pheidas)

Let $n, m \in \mathbb{N}$ with $0 < n \leq m$. Then $n|_p m$ if and only if there exists $a \in \mathbb{F}_p[[t]]$ with $v_t a^p = m$ such that $a^p(t^n - t^m) = t^n t^m (1 - a^{p-1})$.

Proof Sketch.

Rewrite it as $t^{-m} - t^{-n} = a^{-p} - a^{-1}$.
$\Rightarrow$: Say $m = p^s \cdot n$. Then $a = (t^{-p^{s-1} \cdot n} + t^{-p^{s-2} \cdot n} + ... + t^{-n})^{-1}$.
$\Leftarrow$: Let $n = p^r k$ and $m = p^l i$ with $p \nmid k, i$.

1. $t^{-m} - t^{-n} = a^{-p} - a^{-1}$
2. $t^{-n} - t^{-k} = a_1^{-p} - a_1^{-1}$ (By " $\Rightarrow$ ")
3. $t^{-m} - t^{-i} = a_2^{-p} - a_2^{-1}$ (By " $\Rightarrow$ ")

We compute $t^{-i} - t^{-k} = b^p - b$, where $b = a^{-1} + a_1^{-1} - a_2^{-1}$.
Note that RHS has p-divisible valuation. This forces $i = k$.

□

Lemma

Let $n, m, N \in \mathbb{N}$ with $0 < n \leq m < N/3$. Then $n|_p m$ if and only if there exists $\alpha \in \mathbb{F}_p[t]/(t^N)$ such that $\alpha^p(t^n - t^m) = t^n t^m(1 - \alpha^{p-1})$ and $\alpha^{3p} \neq 0$ in $\mathbb{F}_p[t]/(t^N)$.

Proof Sketch.

" $\Rightarrow$ ": As before.

" $\Leftarrow$ ": Let $n = p^r k$ and $m = p^l i$ with $p \nmid k, i$. Let $a \in \mathbb{F}_p[[t]]$ be a lift of α. Note that $v_t a^p < N/3$.

1. We will have by assumption that
   $t^{-m} - t^{-n} = a^{-p} - a^{-1} + t^{N-m-n} z a^{-p}$, for some $z \in \mathbb{F}_p[[t]]$.
2. Set $\epsilon = t^{N-m-n} z \alpha^{-p}$; note that $v_t \epsilon > 0$.
3. Find $a_1, a_2 \in \mathbb{F}_p[[t]]$ such that $t^{-n} - t^{-k} = a_1^{-p} - a_1^{-1}$ and $t^{-m} - t^{-i} = a_2^{-p} - a_2^{-1}$.
4. Deduce that $t^{-i} - t^{-k} = b^p - b + \epsilon$.

Claim: $i = k$. Otherwise, LHS has negative valuation. This forces RHS to have p-divisible valuation. This is contrary to the fact that $p \nmid i, k$. Therefore $i = k$ and $n|_p m$.

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Model-theoretic Fontaine-Wintenberger

Theorem: $G_{\mathbb{Q}_p(p^{1/p^\infty})} \cong G_{\mathbb{F}_p((t))}$. (Fontaine-Wintenberger '79)
Take $K_1 \subsetneq K_2 \subsetneq \ldots$ with $K_n/\mathbb{Q}_p$ totally ramified. Set $K = \prod_{n\in\mathbb{N}} K_n/U$, U non-principal and $K_\infty = \bigcup_{n\in\mathbb{N}} K_n$.

Theorem

K admits a henselian defectless valuation w with $k_w = \mathbb{F}_p((t))$ and Γ_w divisible.

Remark: $G_K/I_w = G_{\mathbb{F}_p((t))}$.
Consider $\mathrm{res} : G_K \twoheadrightarrow G_{K_\infty}$.

Theorem (Model-theoretic Fontaine-Wintenberger)

Let $K_n = \mathbb{Q}_p(p^{1/p^n})$. We then have $Ker(res) = I_w$. In particular, $G_{\mathbb{Q}_p(p^{1/p^\infty})} \cong G_{\mathbb{F}_p((t))}$.

Theorem

K admits a henselian defectless valuation w with $k_w = \mathbb{F}_p((t))$ and Γ_w divisible.

Proof Sketch.

Let v be the natural valuation on K; (K, v) is henselian defectless.

1. $\varpi = \lim_{n \to U} \varpi_n$.
2. Consider $\mathcal{O}_w = \mathcal{O}_v[\frac{1}{\varpi}]$; (K, w) is henselian defectless.
3. $\mathrm{char}(k_w) = p$ since $wp > 0$. $(v\varpi << vp)$
4. Let $\overline{v}$ be the induced valuation on k_w.
5. Use $\aleph_1$-saturation of $\mathcal{O}_v$ to compute that $k_w = \mathbb{F}_p((t))$.
6. $(\Gamma_v, vp) \equiv (\mathbb{Z}, 1)$, so $\Gamma_w = \Gamma_v / \mathbb{Z}vp$ is divisible.

□

Remark: We haven't used wild ramification yet.

Comparison of pictures

$K_\infty = \widehat{\mathbb{Q}_p(p^{1/p^\infty})}$, $K_\infty^\flat = \widehat{\mathbb{F}_p((t))^{1/p^\infty}}$ and $K = \prod \mathbb{Q}_p(p^{1/p^n})/U$.

Classical picture:

$$\begin{array}{ccccc} K_\infty - \text{fét} & \longrightarrow & K_\infty^{\circ a} - \text{fét} & \longrightarrow & (\mathcal{O}_{K_\infty}/p)^a - \text{fét} \\ & & & & \downarrow \cong \\ K_\infty^\flat - \text{fét} & \longrightarrow & K_\infty^{\flat \circ a} - \text{fét} & \longrightarrow & (\mathcal{O}_{K_\infty^\flat}/\pi)^a - \text{fét} \end{array}$$

New picture:

$$K_\infty - \text{fét} \xrightarrow{\cong} K^{\circ w} - \text{fét} \xrightarrow{\cong} \mathbb{F}_p((t)) - \text{fét}$$

Theorem (Fontaine-Wintenberger)

Let $K_n = \mathbb{Q}_p(p^{1/p^n})$ *and* $res : G_K \twoheadrightarrow G_{K_\infty}$ *as before. We then have* $Ker(res) = I_w$*. In particular,* $G_{\mathbb{Q}_p(p^{1/p^\infty})} \cong G_{\mathbb{F}_p((t))}$.

Proof Sketch.

Claim: $K^{I_w} = K \cdot \overline{K_\infty}$.

"$\subseteq$": Say $F/\mathbb{F}_p((t))$ is Galois, the splitting field of $f(x)$. Consider $f_n(x) =$ "$f(x)$ with t replaced by p^{1/p^n}". The splitting fields of the f_n's stabilize to the lifting of F.

"$\supseteq$": If L/K_∞ finite, then $\Omega_{\mathcal{O}_{LK,v}/\mathcal{O}_{K,v}}$ is killed by ϖ.

$$\Omega_{\mathcal{O}_{LK,w}/\mathcal{O}_{L,w}} = \Omega_{\mathcal{O}_{LK,v}/\mathcal{O}_{L,v}}[\frac{1}{\varpi}] = 0$$

(Kähler differentials compatible with extension of scalars).

□

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