The etale-open topology

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Section 1

Review

Etale images

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Definition

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The *etale open* topology $\mathcal{E}_V$ is the topology on $V(K)$ with etale images as the basis.

Examples

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(Examples: $\mathbb{R}, \mathbb{Q}_p, \mathbb{C}((t)), \ldots$).

Theorem

If $K = \mathbb{Q}$, then $\mathcal{E}_V$ is the discrete topology on $V(\mathbb{Q})$.

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2. If $f : V \to W$ is an open immersion, then $(V(K), \mathcal{T}_V) \to (W(K), \mathcal{T}_W)$ is an open embedding.
3. If $f : V \to W$ is a closed immersion, then $(V(K), \mathcal{T}_V) \to (W(K), \mathcal{T}_W)$ is a closed embedding.

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2. The system of discrete topologies $\mathcal{D}_\bullet$ is the finest system of topologies.
3. The system of etale open topologies $\mathcal{E}_\bullet$ is the coarsest system of topologies satisfying

   If $V \to W$ is etale, then $(V(K), \mathcal{E}_V) \to (W(K), \mathcal{E}_W)$ is an open map.

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2. Not all systems of topologies arise this way.
3. A system of topologies $\mathcal{T}_\bullet$ comes from a field topology if and only if this is a homeomorphism for all V, W:

$$((V \times W)(K), \mathcal{T}_{V \times W}) \to (V(K), \mathcal{T}_V) \times (W(K), \mathcal{T}_W).$$

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3. Fractional linear transformations are continuous.

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- $\mathrm{Frac}(\mathbb{C}[[x,y]])$.

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The etale-open topology is interesting iff K is large.

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3. $\mathcal{T}_{Res_{L/K}(V)}$ is a topology on $V(L)$.
4. This defines a system $\mathrm{Ext}_{L/K}\,\mathcal{T}_\bullet$ on L.

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5. If $\mathcal{E}^L_\bullet$ is Hausdorff, totally disconnected, etc., then so is $\operatorname{Ext}_{L/K} \mathcal{E}^K_\bullet$.

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6. If $\mathcal{E}^L_\bullet$ is Hausdorff, totally disconnected, etc., then so is $\mathcal{E}^K_\bullet$.

Section 2

Separation axioms

Hausdorffness

Theorem

The following are equivalent:

1. *K is not separably closed.*
2. *$\mathcal{E}_{\mathbb{A}^1}$ is Hausdorff*
3. *$\mathcal{E}_V$ is Hausdorff for all quasi-projective varieties V.*

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A similar argument handles non-Artin-Schreier closed fields of characteristic p.

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- Some net in $U(K) \cap V(K)$ converges to p in $\mathcal{E}_U$ and to q in $\mathcal{E}_V$.

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Question

Can we always find $f \in K(V)$ such that $f(p), f(q)$ well-defined, and $f(p) \neq f(q)$?

Disconnectedness

Theorem

The following are equivalent:

1. K *is not separably closed and* $K \not\cong \mathbb{R}$.
2. $\mathcal{E}_{\mathbb{A}^1}$ *is totally disconnected.*
3. $\mathcal{E}_V$ *is totally disconnected for all quasi-projective* V/K.

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$$K^p \cap (a + b - K^p) \subseteq \mathbb{A}^1(K).$$

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- A little more work: $\mathbb{P}^n$ is totally disconnected.

Section 3

Pseudofinite fields

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Definition (Ax)

An infinite field K is *pseudo-finite* if

1. K is perfect
2. $\mathrm{Gal}(K) \cong \varprojlim_n \mathbb{Z}/n\mathbb{Z}$.
3. $V(K) \neq \varnothing$ for every geometrically integral V/K.

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Condition (2) means K has a unique degree n Galois extension for all n.

Pseudofinite fields: examples

Example

If $\sigma \in \mathrm{Gal}(\mathbb{Q}^{alg}/\mathbb{Q})$ is random, then the fixed field

$$\{x \in \mathbb{Q}^{alg} : \sigma(x) = x\}$$

is pseudo-finite (with probability 1).

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Most subfields of $\mathbb{F}_p^{alg}$ are pseudo-finite.
e.g., the compositum $\mathbb{F}_{p^2}.\mathbb{F}_{p^3}.\mathbb{F}_{p^5}.\mathbb{F}_{p^7}\cdots$.

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Fact (Ax)

The models of T_{ff} are the finite and pseudo-finite fields.

Etale-open topology on pseudo-finite fields

Let K be pseudo-finite.

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$(\mathbb{A}^2(K), \mathcal{E}_{\mathbb{A}^2}) \to (\mathbb{A}^1(K), \mathcal{E}_{\mathbb{A}^1})^2$ is not a homeomorphism.
$\mathcal{E}_\bullet$ doesn't come from a field topology on K.

Not a field topology

Fact (char$(K) \neq 2$)

The set

$$S = \{(x, y) : x - y \in (K^\times)^2\}$$

is open in $(\mathbb{A}^2(K), \mathcal{E}_{\mathbb{A}^2})$, *but not* $(\mathbb{A}^1(K), \mathcal{E}_{\mathbb{A}^1})^2$.

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Proof sketch.

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Proof sketch.

Suppose $S \supseteq U \times V$ for etale-images $U, V \subseteq K$.
Take distinct $a_1, a_2, a_3, \ldots \in U$. Then $V \subseteq \bigcap_{i=1}^n S_{a_i}$. ($S_a$ = slice over a.)
Use pseudo-finite measure: $\mu(V) > 0$ but

$$\lim_{n \to \infty} \mu\left(\bigcap_{i=1}^n S_{a_i}\right) = 0.$$

□

Etale-open topology on pseudo-finite fields

Using model completeness results, one can prove...

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Similar results hold on many other model-theoretically tame fields.

Section 4

Stable large fields

Stability

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Note: "definable" $=$ "definable with parameters."

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Conjecture (Stable fields conjecture)

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Theorem

If K is stable and large, then K is separably closed.

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5. Can find $a_1, \ldots, a_n \in K^\times$ with

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6. $\{0\}$ is $\mathcal{E}$-open, K is not large.

Fields and classification theory

Stable fields conjecture $\iff$

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Shelah conjecture *implies*

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Any NIP field is finite or large.

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Meta-conjecture

Every model-theoretic conjecture classifying nice fields is equivalent to "All nice fields are large or finite."

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Definition

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Theorem (?)

This defines a system of topologies over K.

The finite-closed topology

Theorem (?)

Let K be a field and $\mathcal{F}_\bullet$ be the system of finite-closed topologies.

1. *$\mathcal{F}_\bullet$ is the coarsest system of topologies such that*

$$U(K) \to V(K)$$

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Corollary (?)

On pseudo-finite, algebraically closed, real closed, and p-adically closed fields, $\mathcal{F}_\bullet = \mathcal{E}_\bullet$.

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- If not, bounded fields are large.
- Even the case of polynomial maps $f : \mathbb{A}^1 \to \mathbb{A}^1$ is unknown.

Reference

Johnson, Tran, Walsberg, Ye. *Etale-open topology and the stable field conjecture*. arXiv:2009.02319.