

# Linearization procedures in the semi-minimal analysis of algebraic differential equations

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# Introduction

The questions I am going to talk about come from two different origins:

- geometric stability theory specialized to the theory  $\mathbf{DCF}_0$  of differentially closed fields of characteristic 0.
- the transcendence properties of the solutions of algebraic differential equations going back to Painlevé.

## Two examples

- The first Painlevé equation  $(E) : y'' = 6y^2 + t$  (Nishioka, Umemura, Nagloo-Pillay)

Painlevé, Umemura, Nishioka	Geometric stability theory
the analytic solutions of $(E)$ are “new” transcendental functions.	the set $\mathcal{S}(E)$ of solutions of $(E)$ in a diff.closed field is <b>strongly minimal</b> .
any tuple of distinct solutions of $(E)$ are, together with their first derivatives, algebraically independent.	the set $\mathcal{S}(E)$ of solutions of $(E)$ in a diff.closed field is a pure infinite set.

- An order one equation  $f(y, y' t) = 0$  “without movable singularities” (Painlevé, Buium).

Painlevé, Umemura, Nishioka	Geometric stability theory
the analytic solutions of $(E)$ can be expressed using <b>classical meromorphic functions</b> such as exponentials, logarithms and elliptic functions.	the set $\mathcal{S}(E)$ of solutions of $(E)$ in a diff.closed field is <b>internal to the constants</b> .

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2 Linearization along an algebraic solution (in dimension 2)

3 Linearization along a generic solution (in dimension 2)

# Autonomous differential equations

I will consider the case of complex autonomous algebraic differential equations:

$$F(y, y', \dots, y^{(n)}) = 0 \text{ where } F \in \mathbb{C}[X_0, \dots, X_n]$$

and focus on the behavior of their generic solutions.

- Such an equation will be presented geometrically (forgetting about the embedding in a jet space) as a pair  $(X, \nu)$  where  $X$  is an (irreducible) algebraic variety and  $\nu$  is a vector field on  $X$ . The vector field  $\nu$  induces a derivation  $\delta_\nu$  on  $\mathbb{C}(X)$  defined by

$$\delta_\nu(f) = df(\nu).$$

- I will write

$$\phi : (X, \nu) \rightarrow (Y, w)$$

to mean that  $\phi$  is a morphism of complex algebraic varieties such that  $d\phi(\nu) = w$ .

We obtain a category  $\mathcal{C}$  called **the category of complex  $D$ -varieties** by Buium.

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We obtain a category  $\mathcal{C}$  called **the category of complex  $D$ -varieties** by Buium.

- (1) This category has (fiber) products: if  $(X, \nu)$  and  $(Y, w)$  are objects of  $\mathcal{C}$ .

$$(X, \nu) \times (Y, w) = (X \times Y, \nu \times w).$$

- (2) A **closed invariant subvariety of  $(X, \nu)$**  is a closed algebraic subvariety  $Z$  of  $(X, \nu)$  which is tangent to the vector field  $\nu$ .

## Solution sets in a differentially closed field

We fix once for all a saturated model  $(\mathcal{U}, \delta_{\mathcal{U}})$  of the theory  $\mathbf{DCF}_0$ . We consider the “functor”

$$(E) : F(y, y', \dots, y^{(n)}) = 0 \rightarrow \mathcal{S}(E) = \begin{cases} \text{set of sol. of } (E) \\ \text{in } (\mathcal{U}, \delta_{\mathcal{U}}) \end{cases}$$

- Starting from the category  $\mathcal{C}$ , we obtain a solution functor:

$$\mathcal{S} : \mathcal{C} \rightarrow \text{Def}(\mathbf{DCF}_0/\mathbb{C})$$

defined by

$$\mathcal{S}(X, \nu) = (X, \nu)^\delta = \{\bar{x} \in X(\mathcal{U}) \mid \delta_{\mathcal{U}}(\bar{x}) = \nu(\bar{x})\}.$$

- Since  $X$  is irreducible, the conditions

$x \in (X, \nu)^\delta$  and  $x \notin Z(\mathcal{U})$  for every proper invariant algebraic subvariety  $Z$  of  $X$

isolate a complete type  $p \in \mathcal{S}(\mathbb{C})$  which will be called the **generic type** of  $(X, \nu)$ .

Analytic geometry	Geometric stability theory
the solutions of $(X, \nu)$ are the analytic curves $t \mapsto x(t)$ on $X(\mathbb{C})^{an}$ tangent to $\nu$ .	the solutions of $(X, \nu)$ are elements of an abstract differential field.
the solutions $t \mapsto x(t)$ which are Zariski-dense in $X$ .	the realizations of the generic type of $(X, \nu)$ .

# The semi-minimal analysis in $\mathbf{DCF}_0$

Let  $(X, v)$  be an autonomous differential equation of positive dimension.

- The set of solutions  $(X, v)^\delta$  is a definable set of finite Morley rank and more precisely:

$$1 \leq MR((X, v)^\delta) \leq \dim(X).$$

- When  $X$  is a curve, then  $(X, v)^\delta$  is always a strongly minimal set and one can use the powerful structure theory for strongly minimal sets in  $\mathbf{DCF}_0$  to describe the structure of autonomous differential equations of dimension one (Hrushovski-Itai).
- I will be interested in the case where  $\dim(X) = 2, 3, \dots$ . In that case,  $(X, v)^\delta$  is only a definable set with “small” finite Morley rank but not always strongly minimal.

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A well-known principle: one can recover information about a definable set  $D$  of finite Morley rank in a stable theory as follows:

- first obtain some information about the strongly minimal sets of the stable theory  $T$  (Zilber’s trichotomy).
- study how  $D$  interacts these with strongly minimal sets (orthogonality, internality, analyzability).

I will refer to this process as the [semi-minimal analysis](#) of the definable set  $D$  (or of a type living on  $D$ ).

**Question:** How to describe this process from the point of view of analytic geometry?

## Non locally modular strongly minimal sets

The most obvious strongly minimal set in the theory  $\mathbf{DCF}_0$  is the field of constants defined by

$$\mathcal{C} = \{x \in \mathcal{U} \mid \delta(x) = 0\}.$$

### Theorem (Hrushovski)

Let  $X$  be a non locally modular strongly minimal set in  $\mathbf{DCF}_0$ . Then  $X$  is *almost internal to the constants*: there exists a definable finite to one map (with parameters)

$$\phi : X \rightarrow W \subset \mathcal{C}^n$$

onto a definable subset  $W$  of  $\mathcal{C}^n$ .

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### Examples:

- (i)  $y' = y$  and  $y' = 1/t$  define non locally modular strongly minimal sets.
- (ii) More generally, any Riccati equation (e.g. over  $\mathbb{C}(t)$ )

$$y' = a(t)y^2 + b(t)y + c(t) \text{ with } a(t), b(t), c(t) \in \mathbb{C}(t).$$

- (iii) any elliptic equation  $(y')^2 = y^3 + ay + b$  where  $a, b \in \mathbb{C}$ .

**Observation:** the solutions of (i), (ii) and (iii) are “classical” meromorphic functions: the **exponential** function, the complex **logarithm** for (i), the solutions of (ii) can be written using a solution of an order 2 linear equation, the solutions of (iii) are **elliptic functions**.

## Analyzability and orthogonality to the constants

Let  $(X, v)$  be an autonomous differential equation. The two following definitions describe two opposite situations for the interaction between  $(X, v)^\delta$  and the field of constants:

### Definition

We say that the generic type of  $(X, v)$  is **analyzable in the constants** if there exists an iterated fibration:

$$(X, v) \dashrightarrow (X_2, v_2) \dashrightarrow \cdots \dashrightarrow (X_r, v_r)$$

where each  $\phi_i : (X_i, v_i) \rightarrow (X_{i+1}, v_{i+1})$  is a dominant rational morphism between autonomous differential equations and

the generic fibre of  $\phi_i^\delta : (X_i, v_i)^\delta \rightarrow (X_{i+1}, v_{i+1})^\delta$  is almost internal to the constants.

### Definition

We say that the generic type of  $(X, v)$  is **orthogonal to the constants** if for every  $(Y, w)$  such that the generic type of  $(Y, w)$  is analyzable in the constants (as above), there are no **proper** closed invariant subvariety

$$Z \subset (X, v) \times (Y, w) = (X \times Y, v \times w)$$

projecting generically on both factors.

# Classical meromorphic functions

Let  $K \subset (\mathcal{M}(U), \frac{d}{dt})$  be a differential subfield of the field of meromorphic functions on a connected open set  $U \subset \mathbb{C}$ .

## Definition (Umemura)

We say that  $K$  is a differential field of **classical meromorphic functions** (or of meromorphic functions of the class  $\mathcal{C}_0$ ) if there exists a tower of differential fields

$$K_0 = \mathbb{C}(t) \subset K_1 \subset \dots \subset K_n = K$$

such that each step of this tower is obtained from the previous one using one of the following operations:

(A1) Solving an **algebraic equation**:  $K_{i+1} = K_i(\xi)$  is generated by a solution  $\xi$  of:

$$\xi^r + b_{r-1}\xi^{r-1} + \dots + b_1\xi + b_0 = 0 \text{ with } b_0, \dots, b_{r-1} \in K_i.$$

(C1) Solving a **linear differential equation** (of arbitrary order):  $K_{i+1} = K_i\langle \xi \rangle$  is generated by a solution  $\xi$  of:

$$y^{(r)} + b_{r-1}y^{(r-1)} + \dots + b_1y = 0 \text{ où } b_1, \dots, b_{r-1} \in K_i.$$

(C2) **Composing** meromorphic functions of  $K_i$  **with an abelian function** : given a lattice  $\Lambda$  of  $\mathbb{C}^r$  such that  $\mathbb{C}^r/\Lambda$  is an abelian variety,  $K_{i+1} = K_i\langle \xi \rangle$  is the differential field generated by a meromorphic function  $\xi$  of the form

$$\xi = \theta \circ \pi \circ (b_1, \dots, b_r) : U \rightarrow \mathbb{C}^r \rightarrow \mathbb{C}^r/\Lambda \rightarrow \mathbb{C}$$

where  $\theta$  is a meromorphic function on  $\mathbb{C}^r/\Lambda$ ,  $\pi : \mathbb{C}^r \rightarrow \mathbb{C}^r/\Lambda$  the canonical projection and  $b_1, \dots, b_r \in K_i$ .

## Analyzability in the constants and solvability in the class $\mathcal{C}_0$

Let  $(X, \nu)$  be an autonomous differential equation. We will require that  $(X, \nu)$  does not admit non-trivial rational integral:

for every rational function  $f \in \mathbb{C}(X)$ ,  $\delta_\nu(f) = 0 \Rightarrow f \in \mathbb{C}$ .

### Theorem (Pillay, Umemura)

The generic type of  $(X, \nu)$  is *analyzable in the constants* if and only if the “generic” analytic solutions of  $(X, \nu)$  belong to the class  $\mathcal{C}_0$ :

*if  $t \mapsto x(t)$  is a Zariski-dense analytic solution of  $(X, \nu)$  and  $f \in \mathbb{C}(X)$ , then the meromorphic function  $t \mapsto f(x(t))$  is in the class  $\mathcal{C}_0$ .*

$\Rightarrow$  is the hard part. It uses the theory of the binding group and the full extent of Kolchin’s Galois theory (for non linear algebraic groups).

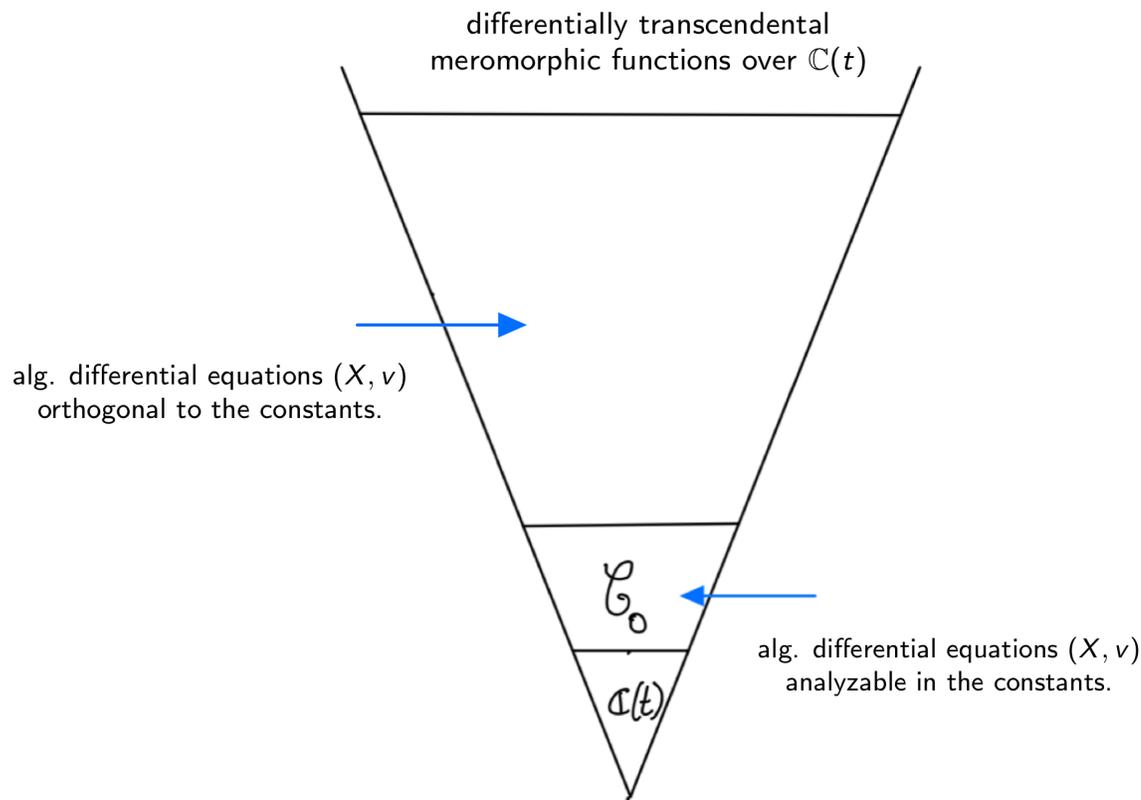
### Corollary

The generic type of  $(X, \nu)$  is *orthogonal to the constants* if and only if the “generic” analytic solutions of  $(X, \nu)$  are alg. ind. over  $\mathbb{C}$  from every meromorphic function in the class  $\mathcal{C}_0$ :

*if  $t \mapsto x(t)$  is a Zariski-dense analytic solution of  $(X, \nu)$  and  $f \in \mathbb{C}(X) \setminus \mathbb{C}$ , then the meromorphic function  $t \mapsto f(x(t))$  is alg ind. from every  $\phi(t) \in \mathcal{C}_0$ .*

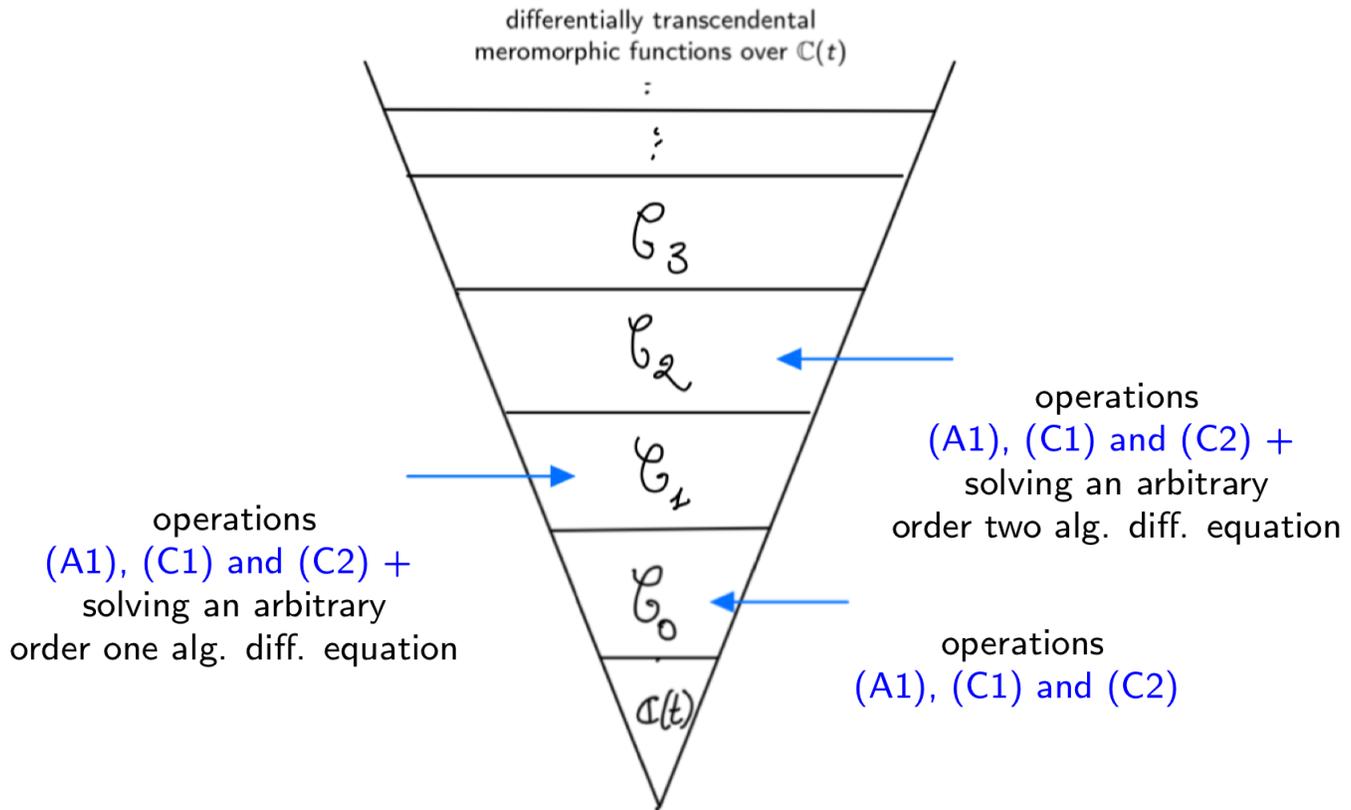
# Painlevé's hierarchy of meromorphic functions

We will represent the differential field  $(\mathcal{M}(U), \frac{d}{dt})$  of meromorphic functions of a connected open set  $U \subset \mathbb{C}$  as follows:



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# The classes $\mathcal{C}_k$ for $k \geq 1$

## Definition (Umemura)

Let  $k \geq 1$ . We say that  $K$  is a differential field of meromorphic functions **in the class  $\mathcal{C}_k$**  if there exists a tower of differential fields

$$K_0 = \mathbb{C}(t) \subset K_1 \subset \dots \subset K_n = K$$

such that each step of this tower using one of the operations **(A1),(C1),(C2)** and

**(P<sub>k</sub>)** Solving an algebraic differential equation **of order  $r \leq k$**  :  $K_{i+1} = K_i\langle \xi \rangle$  is generated by a solution  $\xi$  of:

$$P(y, y', \dots, y^{(r)}) = 0 \text{ where } P \in K_i[X_0, \dots, X_n].$$

## Proposition

Let  $(X, v)$  be an autonomous algebraic differential equation of **dim.  $n > 1$**  without non-trivial rational integral. TFAE:

- (i) the generic type of  $(X, v)$  is **minimal** (has U-rank 1).
- (ii) the analytic solutions of  $(X, v)$  are **new meromorphic functions** (in the sense of Painlevé):  
if  $t \mapsto x(t)$  is a Zariski-dense analytic solution of  $(X, v)$  and  $f \in \mathbb{C}(X) \setminus \mathbb{C}$ , then

$$\text{the meromorphic function } t \mapsto f(x(t)) \in \mathcal{C}_n$$

is algebraically independent (over  $\mathbb{C}$ ) of all meromorphic functions in the class  $\mathcal{C}_{n-1}$ .

# Summary

Let  $(X, \nu)$  be an autonomous differential equation of dimension  $n \geq 2$  without non-trivial rational integral. Denote by  $q$  the generic type of  $(X, \nu)$  and by  $t \mapsto x(t)$  a Zariski-dense analytic solution of  $(X, \nu)$ .

## Analytic formulation

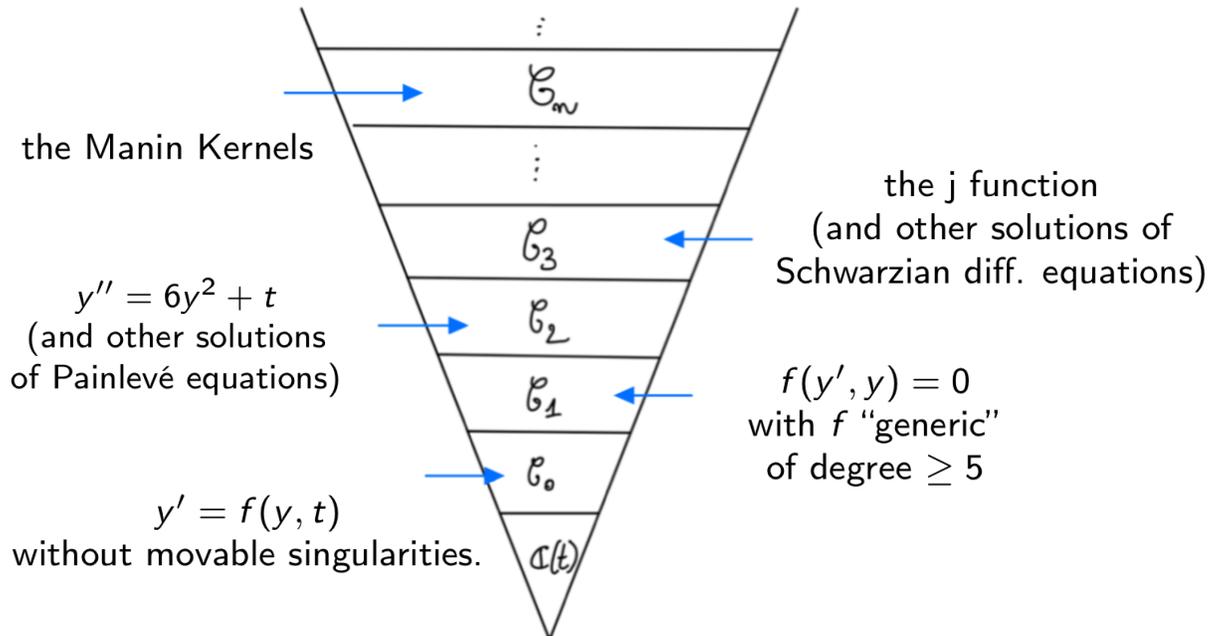
$\forall f \in \mathbb{C}(X) \setminus \mathbb{C}, t \mapsto f(x(t))$  is alg. ind. from all mer. functions of the class  $\mathcal{C}_0$ .

$\forall f \in \mathbb{C}(X) \setminus \mathbb{C}, t \mapsto f(x(t))$  is alg. ind. from all mer. functions of the class  $\mathcal{C}_{n-1}$ .

## Geometric stability theory

$q$  is orthogonal to the constants.

$q$  is minimal.



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## Linearization procedures

By a linearization procedure, I mean an argument of the following form:

autonomous differential equation  $(X, \nu) \rightarrow$  a linear differential equation

such that one can recover information about the generic solutions of  $(X, \nu)$  using information about this linear differential equation and its differential Galois group (or its binding group).

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**An example** (in the analytic setting): Assume that we are able to compute **one particular** analytic solution

$t \mapsto x(t)$  of the differential equation  $(X, \nu)$ .

- If  $y(t)$  is another analytic solution sufficiently close to  $x(t)$  (for small times  $t$ ) then we can write (in a chart)

$$x(t) = y(t) + \epsilon(t)$$

So that

$$\epsilon'(t) = \nu(x(t)) - \nu(y(t)) = d\nu_{x(t)}(\epsilon(t)) + \text{higher order terms (in } \epsilon).$$

- If we abandon the higher order terms, we obtain a linear differential equation

$$\epsilon'(t) = d\nu_{x(t)}(\epsilon(t))$$

called the **linearization of  $(X, \nu)$  along the solution  $x(t)$**  .

**Goal:** study a generic solution of  $(X, \nu)$  using properties of a particular solution and of the linearization along this solution.

# Linear differential equations

Let  $(X, \nu)$  be an autonomous differential equation.

## Definition

A **linear differential equation over  $(X, \nu)$**  of order  $r$  consists of the following data:

- a vector bundle  $\pi : E \rightarrow X$  over  $X$  of rank  $r$ .
- a partial connection  $\nabla_\nu$  on the vector bundle  $E$  along the vector field  $\nu$ .

Such a partial connection can be identified with a vector field  $\nu_E$  on the total space  $E$  such that

$$\pi : (E, \nu_E) \rightarrow (X, \nu) \text{ is a morphism of } \mathcal{C}$$

and the vector field  $\nu_E$  is linear on the fibres.

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**Local picture:** If  $U \subset \mathbb{C}^n$  (or more generally an étale cover of such an open set) and  $E = U \times \mathbb{C}^r$  is the trivial bundle. Then

- $\nu$  has the form  $\nu = f_1 \frac{\partial}{\partial x_1} + \dots + \dots f_n \frac{\partial}{\partial x_n}$  where  $f_i \in \mathcal{O}_X(U)$ .
- In coordinates  $(x_1, \dots, x_n, \epsilon_1, \dots, \epsilon_r)$ ,  $\nu_E$  has the form

$$\nu_E = f_1 \frac{\partial}{\partial x_1} + \dots + \dots f_n \frac{\partial}{\partial x_n} + \left( \sum a_{i,1} \epsilon_i \right) \frac{\partial}{\partial \epsilon_1} + \dots + \left( \sum a_{i,r} \epsilon_i \right) \frac{\partial}{\partial \epsilon_r}$$

where  $a_{i,j} \in \mathcal{O}_X(U)$  is a function on  $U$ .

- The generic fibre of  $\pi$  is a linear differential equation

$$Y' = AY \text{ where } A = (a_{i,j}) \in M_n(\mathbb{C}(X))$$

## Linearization along a closed invariant subvariety

Let  $X$  be a smooth algebraic variety and  $Z$  a smooth closed subvariety of  $X$ . The normal bundle of  $Z$  in  $X$  denoted  $N_{X/Z}$  is defined by the exact sequence of vector bundles on  $Z$ :

$$0 \rightarrow T_Z \rightarrow T_{X|Z} \rightarrow N_{X/Z} \rightarrow 0.$$

### Construction

Assume that  $v$  is a vector field on  $X$  such that  $Z$  is *invariant under  $v$*  (i.e. tangent to the vector field  $v$ ). Then the vector field  $v$  induces:

- a vector field  $v_Z$  obtained by restricting the vector field  $v$  on  $X$  to  $Z$ .
- a partial connection  $\nabla_v$  on  $N_{X/Z}$  along the vector field  $v_Z$ .

The linear equation  $(N_{X/Z}, \nabla_v)$  will be call the (normal) *linearization of  $X$  along the invariant subvariety  $Z$* .

## Linearization along a closed invariant subvariety

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- As described before, after choosing some coordinates, we can identify the generic fibre of  $\pi : (N_{X/Z}, \nabla_v) \rightarrow (Z, v_Z)$  of the form

$$Y' = A.Y \text{ where } A \in M_n(\mathbb{C}(Z)).$$

- The group  $GL_r(\mathbb{C}(Z))$  acts on such differential equation by gauge transformation:

$$A \mapsto BAB^{-1} + B.B^\delta.$$

We denote by  $[N_{X/Z}]$  the equivalence class of the generic fibre modulo gauge transformation.

## Linearization in dimension two

For any differential field  $(K, \delta)$ , we can identify the equivalence classes of linear differential equation of **order one** with the cokernel of

$$d\log : \begin{cases} (K^*, \times) \rightarrow (K, +) \\ x \mapsto \delta(x)/x. \end{cases}$$

### Theorem (with L. Jimenez and A. Pillay)

Let  $(X, \nu)$  be a smooth autonomous differential equation with an invariant algebraic curve  $C$ . Assume that

- (i) the differential equation  $(C, \nu_C)$  is orthogonal to the constants.
- (ii) For all  $n \geq 1$ ,  $n \cdot [(N_{X/Z}, \nabla_\nu)] \notin \mathbb{C} + d\log(\mathbb{C}(C)^*)$

Then the generic type of  $(X, \nu)$  is orthogonal to the constants.

- The condition (ii) expresses that up to gauge transformation over  $\mathbb{C}(C)^{alg}$ , the linear differential equation  $(N_{X/Z}, \nabla_\nu)$  does not descend to the constants.

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Then the generic type of  $(X, \nu)$  is orthogonal to the constants.

- The condition (ii) expresses that up to gauge transformation over  $\mathbb{C}(C)^{alg}$ , the linear differential equation  $(N_{X/Z}, \nabla_\nu)$  does not descend to the constants.
- When  $C$  is a rational curve, the conditions (i) and (ii) can be ensured by computing residues of rational functions on  $\mathbb{P}^1$ . For example

$$(E_1) \begin{cases} y' = yx + g_2(x)y^2 + \dots + g_m(x)y^m \\ x' = x^3(x-1) + f_1(x)y + \dots + f_n(x)y^m \end{cases} \quad \text{and} \quad (E_2) : \begin{cases} y' = yx \\ x' = x^2(x-1) \end{cases}$$

admit  $y = 0$  as an invariant curve and satisfy (i).  $(E_1)$  does satisfy (ii) but not  $(E_2)$ .

# Explanations

- Denote by  $q$  the generic type of  $(X, \nu)$ .  
 $q$  is non-orthogonal to the constants if and only if for  $n \gg 0$ ,  $(X, \nu)^n$  admits a non-trivial (i.e. dominant) rational integral  $f : (X, \nu)^n \rightarrow (\mathbb{A}^1, 0)$ .
- Fix  $n \geq 1$  and consider

$$C_n = C \times \dots \times C \rightarrow X \times \dots \times X.$$

$C_n$  is invariant under the vector field  $\nu \times \dots \times \nu$  and we can consider the linearization of  $(X, \nu)^n$  along the closed invariant subvariety  $C_n$ . We obtain a linear differential equation

$$(E_n) : Y' = A_n Y$$

where  $A_n$  is a matrix of size  $n$  with parameters in  $\mathbb{C}(C_n)$ . We denote by  $G_n$  the Galois group of this linear differential equation.

# Explanations

- Denote by  $q$  the generic type of  $(X, \nu)$ .

$q$  is non-orthogonal to the constants if and only if for  $n \gg 0$ ,  $(X, \nu)^n$  admits a non-trivial (i.e. dominant) rational integral  $f : (X, \nu)^n \rightarrow (\mathbb{A}^1, 0)$ .

- Fix  $n \geq 1$  and consider

$$C_n = C \times \dots \times C \rightarrow X \times \dots \times X.$$

$C_n$  is invariant under the vector field  $\nu \times \dots \times \nu$  and we can consider the linearization of  $(X, \nu)^n$  along the closed invariant subvariety  $C_n$ . We obtain a linear differential equation

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**Observation** : If  $f : (X, \nu)^n \rightarrow (\mathbb{A}^1, 0)$  is a non-trivial rational integral then  $f$  induces a non-trivial rational integral of the linearization  $(E_n)$  of  $(X, \nu)^n$  along  $C_n$ . In particular an **obstruction** to the existence of non-trivial rational integral is

$G_n$  acts transitively on the set of solutions of  $(E_n)$  for all  $n \geq 1$ .

**Conclusion** We want to ensure not only that the “first” Galois group  $G_1 = \mathbb{G}_m$  is as large as possible but also that all the “higher dimensional” ones

$$G_2, G_3, \dots, G_n, \dots$$

are sufficiently large.

## Explanations II

When  $n$  varies, the Galois group  $G_n$  are Galois groups of linear differential equations defined over a varying differential field  $\mathbb{C}(C_n)$ .

- an idea of L. Jimenez PhD is to replace the Galois groups by Galois groupoids. In contrast with the Galois group, the Galois groupoid of  $(E_n)$  will always be definable over  $\mathbb{C}$ .
- In particular, replacing groups by groupoids, Jimenez was able to use the projections  $\pi_i : X^{n+1} \rightarrow X^n$  to obtain “face maps”

$$\pi_{i*} : G_{n+1} \rightarrow G_n \text{ for } i = 1, \dots, (n + 1)$$

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### Lemma

*When  $G_1 = \mathbb{G}_m$  and the curve  $(C, v_C)$  is orthogonal to the constants, the face maps are surjective. Moreover, the following are equivalent*

- (i) *The sequence  $(G_n, n \in \mathbb{N})$  collapses: for  $n \gg 0$ , the face maps*

$$\pi_{i*} : G_{n+1} \rightarrow G_n$$

*have finite kernels. In particular, the sequence  $(\dim(G_n))_{n \in \mathbb{N}}$  is eventually constants.*

- (ii)  $G_2 \neq \mathbb{G}_m \times \mathbb{G}_m$

- (iii) *There is  $n \geq 1$  such that  $n \cdot [(N_{X/Z}, \nabla_v)] \in \mathbb{C} + d \log(\mathbb{C}(C)^*)$*

## Comments

- **In progress:** generalizations to higher dimension (more precisely to higher codimensions). A particularly interesting case for applications:
  - an invariant curve  $C$  in a autonomous diff. equation  $(X, v)$  of dimension 3 such that the generic fibre of the linearization of  $(X, v)$  along  $C$  has Galois group  $SL_2(\mathbb{C})$ .
- Using a perturbation technique, this method can also be applied to certain autonomous differential equations which **do not admit any invariant curve**:

### Corollary

*Let  $\{(X(\alpha), v_\alpha), \alpha \in S(\mathbb{C})\}$  be a smooth family of autonomous differential equation indexed by an (irreducible) algebraic variety  $S$ . If some fixed  $\alpha_0 \in S(\mathbb{C})$ ,  $(X(\alpha_0), v_{\alpha_0})$  satisfies the assumptions of the theorem then the conclusion of the theorem holds for almost all  $\alpha \in S(\mathbb{C})$*

- But this method can not distinguish whether the generic solutions of  $(X, v)$  are **new** meromorphic functions of the class  $\mathcal{C}_2$  or meromorphic functions of the class  $\mathcal{C}_1$ .

**Why?** Because  $(X, v)$  can be equal to its own linearization  $(N_{X/C}, \partial_v)$ ! In that case, the analytic solutions of  $(X, v)$  are always in the class  $\mathcal{C}_1$ .

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- 1 How to describe a definable set of  $\text{DCF}_0$  using analytic solutions?
- 2 Linearization along an algebraic solution (in dimension 2)
- 3 Linearization along a generic solution (in dimension 2)

## Linearization at the generic point

Let  $(X, v)$  be an autonomous differential equation and  $\pi : T_X \rightarrow X$  the tangent bundle of  $X$ .

### Construction

The vector field  $v$  can be extended to a partial connection  $\nabla_v$  on  $T_X$  along  $v$  so that

$$\pi : (T_X, \nabla_v) \rightarrow (X, v) \text{ is a morphism of } \mathcal{C}.$$

It is characterized by the property that if  $f \in \mathbb{C}(X)$  then

$$\delta_{\nabla_v}(df) = d(\delta_v(f)) \text{ where } df \text{ is identified with a function on } T_X.$$

The linear differential equation  $(T_X, \nabla_v)$  will be called the **linearization of  $(X, v)$  at the generic point**.

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The linear differential equation  $(T_X, \nabla_v)$  will be called the **linearization of  $(X, v)$  at the generic point**.

- I will describe the case where  $\dim(X) = 2$ . In that case, the generic fibre of

$$\pi : (T_X, \nabla_v) \rightarrow (X, v)$$

is a linear differential equation of order 2 with parameters in  $\mathbb{C}(X)$ .

- We will denote by  $G(X, v)$  its Galois group which can be identified with an algebraic subgroup of  $GL_2(\mathbb{C})$ . The vector field  $v$  defines tautological section

$$v : (X, v) \rightarrow (T_X, \nabla_v) \text{ compatible with the diff. structure.}$$

- The Galois group  $G(X, v)$  has to fix this section, so that  $G(X, v)$  is always a subgroup of  $Aff_2(\mathbb{C})$ .

## Linearization at the generic point (dimension two)

When this Galois group  $G(X, \nu)$  is maximal (i.e.  $G(X, \nu) = \text{Aff}_2(\mathbb{C})$ ), we obtain:

### Theorem (J.)

Let  $(X, \nu)$  be an autonomous differential equation of dimension two. Assume that

- (i) the Galois group  $G(X, \nu)$  of the generic fibre of  $\pi : (T_X, \nabla_\nu) \rightarrow (X, \nu)$  is the *affine group*.
- (ii) the generic type of  $(X, \nu)$  is *orthogonal to the constants*.

Then the generic type of  $(X, \nu)$  is *minimal*: if  $t \mapsto x(t)$  is a Zariski-dense analytic solution of  $(X, \nu)$  and  $f \in \mathbb{C}(X) \setminus \mathbb{C}$  then

$t \mapsto f(x(t))$  is a *new* meromorphic function of the class  $\mathcal{C}_2$  (i.e. algebraically independent over  $\mathbb{C}$  of every function  $\phi(t) \in \mathcal{C}_1$ ).

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### In practice,

- To apply this theorem in practice, I showed that if  $\nu$  admits an (isolated) complex zero which is not contained in any closed invariant algebraic curve then condition (i) holds.
- To obtain (ii), we can apply the linearization theorem in dimension two that I described before.

# Explanations

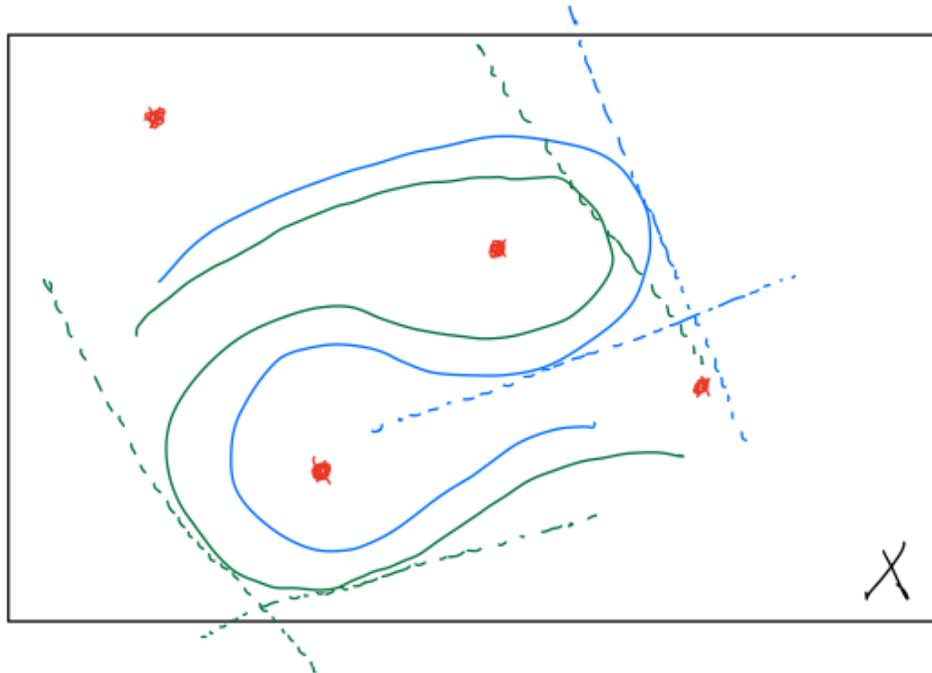
The first tool both in the proof of this theorem and to compute the Galois group  $G(X, \nu)$  in practice are the notions of **invariant foliations** and **invariant webs (of foliations)**.

- The foliations are the rational sections  $\sigma_{\mathcal{F}} : X \dashrightarrow T_X$  of the projectivization

$$\pi : \mathbb{P}(T_X) \rightarrow X$$

of the tangent bundle of  $X$  and the  $d$ -webs are the algebraic sections of degree  $d$  of  $\pi$ .

- Such a foliation  $\mathcal{F}$  defines **outside of its singular locus** a partition of  $X(\mathbb{C})^{an}$  in analytic Riemann surfaces immersed in  $X(\mathbb{C})^{an}$  which are tangent at every point to the field of lines defined by  $\mathcal{F}$ .



# Explanations

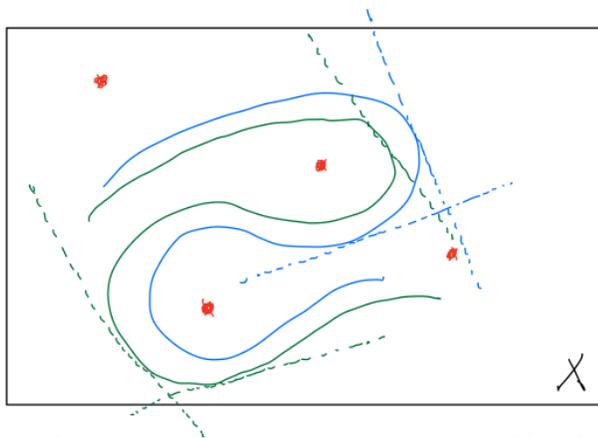
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- We say that a foliation is **invariant under the vector field  $\nu$**  if this partition is preserved by the solutions of the differential equation  $(X, \nu)$ :
  - if  $t \mapsto x(t)$  and  $t \mapsto y(t)$  are two analytic solutions of  $(X, \nu)$  and  $x(0)$  and  $y(0)$  are in the same  $\mathcal{F}$ -leaves then so are  $x(t)$  and  $y(t)$  for all times  $t$  where both solutions are defined.
- In particular, the foliation  $\mathcal{F}(\nu)$  tangent to  $\nu$  is always invariant.

# Explanations II

The Galois group  $G(X, \nu)$  and its connected component  $G^0$  encode some information about the structure of the foliations and webs on  $X$  invariant under  $\nu$ :

## Proposition

*Let  $(X, \nu)$  be an autonomous differential equation. Exactly one of the three following cases occur:*

- (1)  $G(X, \nu) = G^0 = \text{Aff}_2(\mathbb{C})$  if and only if apart from  $\mathcal{F}(\nu)$ , there is no other invariant foliation or any other invariant (irreducible) web.*
- (2)  $G^0 = \mathbb{G}_m(\mathbb{C})$  if and only if there are exactly two irreducible webs invariant under  $\nu$ . One of them is  $\mathcal{F}(\nu)$  and the other one  $\mathcal{W}$  is:
  - (2a) either a foliation,*
  - (2b) or a 2-web.**
- (3)  $G^0 = 0$  if and only if  $\nu$  admits at least three invariant irreducible webs if and only if  $\nu$  admits infinitely many.*

# Explanations II

The Galois group  $G(X, v)$  and its connected component  $G^0$  encode some information about the structure of the foliations and webs on  $X$  invariant under  $v$ :

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- The second ingredient of the proof: the notion of **modularity** (or one-basedness) in geometric stability theory. We use our hypothesis (ii) to show that if the generic type of  $(X, v)$  is not modular then we can produce an invariant foliation  $\mathcal{F} \neq \mathcal{F}(v)$ .
- Once we know the type is one-based, we use the good properties of the semi-minimal analysis of one-based types to produce an irreducible invariant web  $W \neq \mathcal{F}(v)$  when the type is not minimal.

# An application: planar algebraic vector fields

## Theorem (J.)

Consider a differential equation of the form:

$$(E_v) : \begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases} \quad \text{with } f(x, y), g(x, y) \in \mathbb{C}[x, y]$$

Assume that  $f(x, y)$  and  $g(x, y)$  are polynomials of degree  $d \geq 3$  and that the coefficients of  $f(x, y)$  and  $g(x, y)$  satisfy no non-trivial algebraic relation with coefficients in  $\mathbb{Q}$  (in particular, they are all non zero). Then the set of solutions of  $(E_v)$  is:

- (A) **strongly minimal** : the coordinates of every non-constant analytic solutions are new meromorphic functions of the class  $\mathcal{C}_2$ .
- (B) **disintegrated** (or geometrically trivial): if  $(x_1(t), y_1(t)), \dots, (x_r(t), y_r(t))$  are  $r$  non-constant analytic solutions which are not algebraically independent then

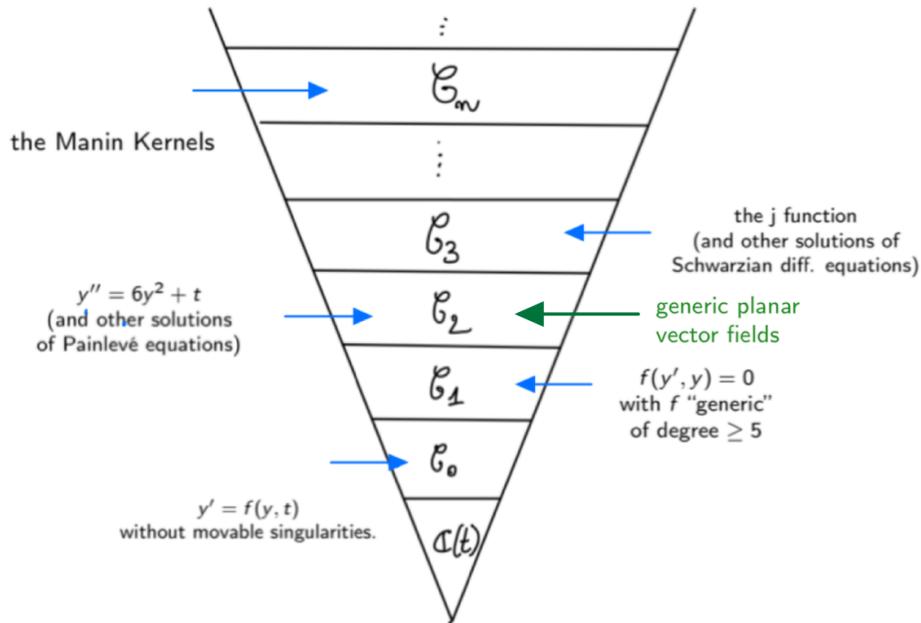
$t \mapsto (x_1(t), y_1(t), x_2(t), \dots, x_r(t), y_r(t))$  is not Zariski-dense in  $\mathbb{C}^{2n}$

then there are  $i \neq j$  such that:

$t \mapsto (x_i(t), y_i(t), x_j(t), y_j(t))$  is not Zariski-dense in  $\mathbb{C}^4$ .

# Generic differential equations

(Poizat 1980): à propos des corps différentiels, on est souvent amené à faire des conjectures dont on est persuadé qu'elles ne peuvent être fausses que pour des équations très particulières et que pourtant on arrive à montrer que dans des cas encore plus particuliers.



- (Shelah 73'): A "sufficiently general" algebraic differential equation of order  $n \geq 2$  is minimal.
- In other words, the generic solutions of such equations are new meromorphic functions of the class  $\mathcal{C}_n$ .

Thank you for your attention