Motivic Poisson summation

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1. Local motivic integration.

2. Deligne-Kazhdan-Vigneras; statement of problem.

3. Adeles. The quotients $T(\mathcal{O})\backslash T(\mathbb{A})/T(K)$.

4. Adelic volumes (integrals of global test functions.)

5. The sum-over-rational-points functional δ^K.

Poisson summation formula: $\delta^K \mathcal{F} = \delta^K$.

6. Motivic covolume and rational points on orbits.

7. Form-independence of Fourier transform.

0. Grothendieck rings.

Let T be a theory in a language L. By a constructible set X we mean here a quantifier-free formula $\phi(x)$ of L, viewed as a set $X(M) = \phi(M)$ in a model M of T. If we wish to speak of formulas with parameters from $A \leq M \models T$, we refer to T_A. We assume substructures are closed under constructible bijections. $[X]$ is the class of X, up to constructible bijections.

$K^+(T) = \{[X] : X \text{ constructible }\}$;

$$[X] + [Y] = [X \dot{\cup} Y],\ [X \cdot Y] = [X \times Y]$$

$\mathcal{K}(T)$ is the ring formed by adding formal additive inverses.

We will also add multiplicative inverses for certain X such that $X(A) \neq \emptyset$ for all A. We think of an element of $\mathcal{K}(T)$ as a generalized number.

If $A \leq M \models T$, A finite, we obtain a homomorphism

$$\mathcal{K}(T) \to \mathbb{Q}, \qquad [X] \mapsto X(A)$$

Summation.

Let X be a constructible set. If $a \in X(M), M \models T$, let $T_a = Th(M,a)$. We have an X-parameterized family of rings $\mathcal{K}(T_a)$.

One can similarly define a Grothendieck ring of constructible sets Y *over* a given constructible set X; we denote it $Fn(X, \mathcal{K})$ and view it as a ring of sections

$$a \mapsto [Y(a)] \in \mathcal{K}(T_a)$$

We have an additive map $Fn(X, \mathcal{K}) \to \mathcal{K}(T)$,

$$Y \mapsto \sum_{x \in X} Y(x) := [Y]$$

This should be viewed as part of the structure of the Grothendieck ring.

Quotients.

Let $\widehat{T}$ be a universal theory extending $T_\forall$.

$$\mathcal{K}(T/\widehat{T}) := \mathcal{K}(T)/(\{[X] : \widehat{T} \models (\neg\exists x)(x \in X)\})$$

Example

ACF_F=theory of algebraically closed fields with an F-algebra structure. If $char(F) = 0$, $\mathcal{K}(ACF_F) = \mathcal{K}(Var_F)$.

For the main theorem we will permit multiplicative inverses for the classes of all abelian algebraic groups; details below.)

Let D be a quantifier-free piecewise-constructible (=strict Ind-constructible) k-algebra. For each field F we have $D(F)$.

The set of F such that $D(F)$ has no 0-divisors is closed under ultraproducts and substructures, so it is the set of models of a universal theory $Div(D)$. We will work with the quotient $\mathcal{K}(ACF_F, /Div(D))$. *A posteriori our main theorem will be valid in* $\mathcal{K}(ACF)_{\mathbb{Q}}$. (Explained below.)

$\mathcal{K}$ is the result of these two operations, plus the provision for an additive character ψ:

(i) Motivic exponential sums $\sum \phi(x)\psi(x)$.

Needed for the definition of the Fourier transform.

Let $\mathcal{K} = ACF$, Let H be a constructible group, $\mathcal{K}_H^0 = Fn(H, \mathcal{K})$, Then $\mathcal{K}_H$ admits a *convolution*: $f * g(a) = \sum_{b \in H} f(a)g(a^{-1}b)$. The characteristic function χ_1 is the identity element. Let $\mathcal{K}_H = \mathcal{K}_H[\chi_c - 1]$ where $c \neq 1$ is some constructible element. For $\phi \in Fn(H, \mathcal{K})$, define $\sum_{x \in H} \phi(x)\psi(x)$ to be the image of ϕ in $\mathcal{K}_H$. Note:

$$\sum_{x \in H} \psi(x) = 0$$

We will use the case $H = G_a$

(ii): Localizing by group classes

In part (6) of this talk we will need to localize by the multiplicative subset of all commutative group varieties. Instead of constructing this localization we note and impose a consequence:

Let $(A_y : y \in Y)$ *be a definable family of commutative algebraic groups,* $(X_y), (X'_y)$ *two families of definable sets, and assume:*

$$[A_y][X_y] = [X_y]^2, \quad [A_y][X'_y] = [X'_y]^2$$

$$[X_y]^2 = [X_y][X'_y] = [X'_y]^2$$

for $y \in Y$*. Then*

$$\sum_{y \in Y} [X_y] = \sum_{y \in Y} [X'_y]$$

.

Let $K(ACF_F)_g$ be the quotient of $K(ACF_F)$ obtained by imposing the above relation, as well as the relations: $a = b$ whenever $[a][A] = [b][A]$.

The relation $[A][X] = [X]^2$ is typical of principal homogeneous spaces X_y.

Proof: Let $e_y = \frac{[X_y]}{[A_y]}, e'_y = \frac{[X'_y]}{[A_y]}$. Then $e_y = e_y^2 = e_y e'_y = (e'_y)^2 = e'_y$. So

$$\sum_{y \in Y} [X_y] = \sum_{y \in Y} e_y [A_y] = \sum_{y \in Y} e'_y [A_y] = \sum_{y \in Y} [X'_y]$$

(iii) Avoiding splitting.

Let $\mathfrak{f}$ be a field, $\mathfrak{l} = \mathfrak{f}$ a cyclic extension of order p. We will be interested in certain infinite dimensional division algebras D over $\mathfrak{f}$; our theorem becomes trivial upon base change to l.

Let Y be a finite variety, such that $\mathfrak{l} = \mathfrak{f}(Y)$. We will work in $\mathcal{K}' = \mathcal{K}(ACF_{\mathfrak{f}})/[Y]$.

Note that $[Y]^2 = p[Y]$, and so $e = [Y]/p$ is idempotent in $\mathcal{K}_{\mathbb{Q}}$. Since our theorem is true (trivially) in $\mathcal{K}/(1-e)$, if we prove an identity in $\mathcal{K}'$ it will be true in $\mathcal{K}_{\mathbb{Q}}$.

Assume $V \leq V'$ are varieties, such that whenever $V' \setminus V$ has a point in a field F, we have $\mathfrak{l} \leq F$. Then $[V] = [V'] \in \mathcal{K}'$. It will thus suffice to consider fields F such that D_F is a division ring. This will allow us to consider certain $\bigvee$-definable sets as definable.

1. Local Integration (of very smooth functions)

$$L((s)) = \{\sum a_i s^i : a_i \in L, a_n = 0 \text{ for } n << 0\}$$

$V_{N,M} = s^{-N}L[[s]]/s^M L[[s]]$, a finite-dimensional L-space (with basis.)

A *local test function* $= \phi \in Fn(V_{n,m}, \mathcal{K})$; viewed as a function on $L((s))$, supported on $s^{-n}L[[s]]$, and locally constant modulo $s^M L[[s]]$. Similarly for several variables.

Integration.

$$\int \phi = [\mathbf{k}]^{-M} \sum_{x \in V_{N,M}} \phi(x)$$

Convolution:

$$(\phi_1 * \phi_2)(x) = \int \phi_1(u)\phi_2(u^{-1}v)$$

Fourier transform: fix a linear $r : D \to \mathbf{k}$ vanishing on $s^{-M}L[[s]]$ for some M; with $\mathcal{O}^\perp = \{x : (\forall y \in \mathcal{O}) r(xy) = 0\} = s^{2\nu}\mathcal{O}$.

$$\mathcal{F}(\phi)(x) = [\mathbf{k}]^{-\nu} \int_y \phi(y)\psi(r(xy))$$

If ϕ is defined modulo $s^M L[[s]]$, then $\mathcal{F}(\phi)$ is supported on $s^{-M}L[[s]]$.

The smooth integration above applies equally when $L[[s]]$ is non-commutative.

Let $L = (L, \sigma)$ be a difference field. Form $L[s]$ with $sa = \sigma(a)s$.

Example: (Manin's quantum plane) $L = k[u], \sigma(u) = qu$. Obtain $k[u, s], su = qus$

$D_0 = L[[s]], L((s))$; a division ring. $PD_0 := D_0^*/Z^*$. $\mathcal{O} = L[[s]]$.

Center $= Z = F((t))$ where $F = Fix(\sigma), t = s^n$. For each irreducible $P[X] \in F[X]$ in one variable, $\{a \in D : F(a) = 0\}$ is either empty or a conjugacy class of D^*.

Example: Given a non-commutative polynomial $g(X_1, \ldots, X_k)$ over L, obtain a power series $P_g(t) = \sum [W_n] t^n \in \mathcal{K}(\text{Diff.-Var})[[t]]$. $W_n = [\{x \in L[[s]]/s^n : g(x) = 0 \mod s^n\}]$.

We will work with $\sigma^n = 1$ on L; and in this talk will assume n is prime.

Then $L((s))$ is an algebra of dimension n^2 over the center $k_n((s^n))$. Convolution of test functions, and Fourier transform can be understood via n^2-dimensional, commutative motivic integration. The non-commutative viewpoint will be used at one point in the proof (to show that $G(\mathcal{O})G(F) = G(\mathbb{A})$ for $G = PsD^*$ via the Euclidean algorithm in D.), but in general we will take the n^2-dimensional, commutative view.

2. Deligne-Kazhdan-Vigneras.

Let $[L : F]$ be a cyclic Galois extension, $[L : F] = n$, and let σ, σ' be two choices of a generator. Form $D, D' = L((s))$ as above. They have center $Z = F((t))$. Let Y be the set of irreducible central polynomials over Z, of degree 1 or n. For $y \in Y$ let $C_y = \{d \in D : y(d) = 0\}$, similarly C'_y.

Can form a character table $Y \times IrrRepPD \to \mathbb{C}$, namely $tr\rho(c)$ where $f(c) = 0$.) Similarly for D'; we have a bijection between the columns (conjugacy classes.)

Theorem. [DKV] p a prime power, $L = \mathbb{F}_{p^n}$. There exists a bijection $Rep(PD) \to Rep(PD')$ respecting the character table.

Equivalent formulation: The identification of conjugacy classes induces an isomorphism of convolution algebras. I.e. let $f_1, f_2, f_3 \in Y$, fix m and c with $f_3(c) = 0$. Let χ_i be the characteristic function of $x : val f_i(x) \geq n\}$. Let $C_m(f_1, f_2, f_3) = \chi_1 * \chi_2(c) = vol(\{x : \mathrm{val} f_1(x) \geq n, \mathrm{val} f_2(c^{-1}x) \geq m\})$. Then $C(f_1, f_2, f_3, m)$ is the same for D, D'.

Nearly equivalent: table for Fourier transform, $\mathcal{F}(\chi_1)(c)$ does not depend on choice of σ.

[DKV] obtain this by comparing both division rings to GL_n; we will not consider this here.

Analogs for other groups are known; cf. Waldspurger.

No local proof is known for any such result.

Theorem 1. *Let n be prime, L any field, $\sigma, \sigma' \in Aut(L)$ with $\sigma^n = 1$. Let $\mathfrak{f} = Fix(\sigma) = Fix(\sigma')$. The Fourier transform table with values in $\mathcal{K}$ is the same for σ, σ'.*

The proof is commutative, global and motivic.

3a. T_{loc}, theory of valued fields valued fields with a section i of res

with sort $K, \mathbf{k}, \Gamma$ for the valued field, value group, residue field.

$$(K, +, \cdot; \Gamma, +, <, 0, 1; \text{val} : VF^* \to \Gamma; \text{res}(\frac{x}{y}); i : \text{res} \to V$$

T_{loc}: K is an algebraically closed field, val is a valuation, with valuation ring $\mathcal{O}$ and maximal ideal $\mathcal{M}$, and $\mathcal{M} \oplus k = \mathcal{O}$.

Delon - Leloup.

T_{loc} admits quantifier elimination. k, Γ are embedded, stably embedded and strongly orthogonal. Note $(K, i(\mathbf{k}), +, \cdot)$ has no QE.

Example: a small neighborhood of $\mathbf{k}$ in K.

3b. Theory of valued fields over a curve

We describe here a first-order theory $\mathbf{T} = T^{gl}$ convenient as the background for adelic work. It has the following sorts.

k - an algebraically closed field with a distinguished field of constants F. k is endowed with the language of F-algebras.

$C(k)$, where C is a smooth, complete curve over F.

Γ - an ordered Abelian group, with distinguished element $1 > 0$.

VF. This sort comes with a map $VF \to C(k)$; the fibers are denoted K_x. Each K_x comes with valuation ring $\mathcal{O}_x$, a surjective homomorphism $\text{res}_x : \mathcal{O}_x \to k$, and a ring embedding $i_x : k \to \mathcal{O}_x$, such that $\text{res}_x \circ i_x = Id_k$. Also, a map

$v_x : K_x \setminus \{0\} \to \Gamma$, denoting a valuation with valuation ring $\mathcal{O}_x$.

We identify k with its image $i_x(k)$. .

As a final element of structure, we have a function $c : C(k) \to VF$, such that $c(x) \in C(K_x)$; and for any $f \in k(C)$, val$f(c(x)) = ord_x(f) \cdot 1$. So for any limited subset S of $k(C)$, an image of S in K_x is definable.

$\mathbf{T}$ admits quantifier-elimination. k and Γ are embedded and stably embedded.

All $\mathbf{T}_C$ are interpreted in $\mathbf{T}_{\mathbb{P}1}$. We will work in $\mathbf{T}_{\mathbb{P}1}$, but will need other $\mathbf{T}_C$ when analyzing some structures definable there.

Over $\mathbf{T}_C$, we form:

$K = k(C)$ as an piecewise definable field

We view $K = (K, +, \cdot, \mathbf{k}, \text{res}(dx))$ as Ind-definable in $\mathbf{T}$. For $C = \mathbb{P}^1$, the pieces are the rational functions of degree $\leq d$.

Consider the integral k-adeles: $\mathcal{O} = \prod_{x \in C(k)} \mathcal{O}_x$.

Adeles $\mathbb{A}$, a subring of $\prod_{x \in C(k)} K_x$.

We have a diagonal embedding $K \to \mathbb{A}$.

3. $T(\mathcal{O})\backslash T(\mathbb{A})/T(K)$

Let X be a definable set. By a *definable function* $f : X \to V(\prod_{c \in C} K_c)$ we mean a definable function f on $X \times C$, such that $f(x,c) \in V(K_c)$.

$f : X \to V(\mathbb{A})$ means: For all x, for all but finitely many c, $f(x,c) \in V(\mathcal{O}_c)$.

Let E be an Ind-definable-in-definable families-equivalence relation on $V(\mathbb{A})$. a *representative constructible set* for V/E to be a definable set Y, such that for some definable X and $f : X \to V(\mathbb{A})$ and surjective $g : X \to Y$, every element of $V(\mathbb{A})$ is E-equivalent to some element $g(x)$, and $g(x) = g(x')$ iff $f(x) E f(x')$.

If Y, Y' are representative constructible sets for $V(\mathbb{A})/E$, there exists a constructible bijection

$Y \to Y'$: $g(x) \mapsto g'(x')$. Define $[V(\mathbb{A})/E] = [Y]$.

Let T be a torus over $F(C)$, and assume given $T(\mathcal{O})$. For each $v \in C(\mathbf{k})$ we have $h_v : T \to X_*(T) \otimes \Gamma$. Let $h = \sum_v h_v$, and $T(\mathbb{A})^0 = \ker(h)$.

We assume $T(\mathcal{O}_v) = \ker(h_v)$ for almost all v.

Lemma 2. *There exists a representative set for $T(\mathcal{O})\backslash T(\mathbb{A})/T(K)$. It admits a definable map to Γ; the kernel, $T(\mathcal{O})\backslash T(\mathbb{A})^0/T(K)$, is an algebraic group.*

The classes $[T(\mathcal{O})\backslash T(\mathbb{A})^0/T(K)$ will play an essential role.

Note: the Adeles are *not* pro-definable or *-definable in the usual sense.

The construction we need is not Pro(Def), but (Def-Pro)(Def). For instance consider $J = G_m(\mathcal{O}) \backslash G_m(\mathbb{A})^0 / G_m(K)$. If we take the Ind/Pro interpretation, every bijection becomes an isomorphism.

4a. Semi-local volumes.

Semi-local test functions: Consider $\prod_{v\in S}\mathbb{A}_v$.

Let $S\subset C(\mathbf{k})$ be finite, definable. Let t_v be a parameter for K_v. It is natural to define semi-local test functions as definable functions, $\prod_v t_v^M\mathcal{O}_v$-invariant for some M, and supported on $\prod_v t_v^{-M}\mathcal{O}_v$. Iterated integration.

In particular, for a product $X=\prod_{v\in S}X_v$, $X_v\subseteq\mathcal{O}_v^m$.

$$\mathrm{vol}(X)=\prod_{v\in S}\mathrm{vol}(X_v)$$

Objections:

i) $\mathrm{vol}(X_v)$ belong to different rings!

cf. Weil reduction of scalars.

ii) Assume S is a Galois orbit. Classically, one takes only *one* copy of $\mathbb{A}_v$. Then expect: $\mathrm{vol}(X \dot\cup Y) = \mathrm{vol}(X) + \mathrm{vol}(Y)$.

cf. Frobenius.

Let $\rho = \rho_F : \mathcal{K} \to \mathcal{K}(ACF_{\mathfrak{f}}, Th_{\forall}(\mathfrak{f}))$, $[X] \mapsto X(\mathfrak{f})$. (If $\mathfrak{f}$ is a finite field, this is the "counting rational points" map.)

$$\rho \prod_{v \in S} (a_v + b_v) = \rho \prod_{v \in S} a_v + \rho \prod_{v \in S} b_v$$

(A separable descent analogue of $(a+b)^p = a^p + b^p$?)

4b. By a *global test function* ϕ we mean a semi-local test function ϕ_S at a finite definable subset $S \subseteq C$, extended by $1_{\mathcal{O}_v}$ to all $v \notin S$. I.e. $\phi(a) = \phi_S(a)$ if $a_v \in \mathcal{O}_v$ for $v \notin S$, otherwise $\phi(a) = 0$.

So for test functions, global integration reduces to the semi-local case.

5. Rational points and Poisson summation.

Let ϕ be a global test function.

The set of points of $K = k(C)$ in the support of ϕ is a limited set, i.e. contained in one of the definable approximations K_n to K.

Define $\delta^K(\phi) = \sum_{a \in K_n} \phi(a)$.

- $\delta^K(\psi(r(ax))\phi(x)) = \delta^K(\phi)$ for $a \in K$.
- $\delta^K(\phi(ax) = \delta^K(\phi)$ for $a \in K$.
- $\delta^K(1_{\mathcal{O}}) = [\mathbf{k}]$

Poisson summation formula:

$$\delta^K \mathcal{F} = \delta^K \quad (1)$$

6. Rational points on orbits and motivic co-volume.

Define integers R_v of D_v, uniformly in $v \in C$. For almost all places R_v is a maximal stably dominated subring of D_v. At the ramified places $0, \infty$ we take $R_0 = D_0$, nothing that modulo the center D_0^* is $Div(D)$-equivalent to a stably dominated group.

$R = \prod_v R_v$. When $C = \mathbb{P}^1$ we have:

$PD(R)PD(K) = PD(\mathbb{A})$.

For $T \leq D^*$, set $T(\mathcal{O}) = T \cap R_v^*$.

Let $c \in D(F)$. Let O be the orbit of c under R^*-conjugation, interpreted geometrically, and let $T = C_{D^*}(c)$. Then

(2) $$\delta^K(1_O) = n[T(\mathcal{O})\backslash T(\mathbb{A})/T(K)][L^*/G_m]$$

Here L is is viewed as a definable ring, with group of units L^*; so $[L^*/G_m]$ is directly the class of a finite-dimensional variety. This is the generic case; when O meets the centralizer of an element normalizing L, this multiplier needs to be modified slightly.

Proof of Theorem 1.

We work over $\mathbb{P}^1$; we define R_v for $v \neq 0, 1$ in such a way that in $ACVF_{F_v}$, (D_v, R_v) is a form of $(M_n, M_n(\mathcal{O})$; they are isomorphic with parameters.

Given any $c \in D(F)$, we show existence of $c' \in D'(F)$ such that (D_v, R_v, c) and (D'_v, R_v, c') are isomorphic with parameters. The family of isomorphisms is a torsor for $T(R_v)$, which acts trivially on T. They induce a unique, hence definable, isomorphism $T(c) \to T(c')$, preserving $T(\mathcal{O})$.

Let ϕ be a local test function at 0. Let O be the conjugacy class of c. Let N_v be a standard neighborhood of c at v, with N_1 small compared to others. $N = \prod_{v \neq 0} N_v$.

" K is discrete in $\mathbb{A}$ "

$$\phi(c) = \frac{\delta^K(\phi \frown 1_N)}{\delta^K(1_N 1_O)} \tag{3}$$

Writing this for $\mathcal{F}(\phi)$ and applying Poisson summation we obtain:

$$\mathcal{F}\phi(c) = \frac{\delta^K(\phi \frown \mathcal{F}(1_N))}{\delta^K(1_N \mathcal{F}^{-1} 1_O)}$$

But by (2) and the isomorphism $T \to T'$, the right hand side is independent of the form.

Problems.

1) (Noncommutative integration.)

Recall $\xi_n : \mathcal{K}(Diff_v ar_F) \to \mathcal{K}(Var_F)$, $[X] \mapsto [X(L \otimes_F k)]$.

a) Assume $\sigma^n(x) = 1$ on L, $F = Fix(\sigma) \cap L$. Then ξ_n depends on the difference field structure of L.

Let $\rho : \mathcal{K}(Var_F) \to \mathcal{K}(Var_{F^a})$ be the natural homomorphism. Then $\rho \circ \xi_n([X]) = [X \cap Fix(\sigma)]$. In particular, $\rho \circ \xi_n(X)$ does not depend on the difference field structure of L. Theorem 1 can be stated as saying that $\rho(X)$ does not depend on L, where X is a class in the Fourier table of D. Give a more general criterion for X to be absolute in this sense.

b) Given a non-commutative polynomial $g(X_1, \ldots, X_k)$ over L, obtain a power series $P_g(t) = \sum [W_n] t^n \in \mathcal{K}$(Diff.-Var)$[[t]]$. $W_n = [\{x \in L[[s]]/s^n : g(x) = 0 \mod s^n\}]$. For any n, $\xi_n(P_g)$ is rational. A statement about P_g implying this? In quantum plane picture, variation of q?

c) Find local, non-commutative (difference variety) proof by studying variation of a finite dimensional qunantity depending on a transformally transcendental q, specializing at $q^n = 1$ to $V \cap \sigma^n = 1$. (E.g. $V \cap \sigma^n(s) = q^n s$?)

d) Change of variable.

2) Local motivic integration on valued fields with a field of representatives.

Every formula of VF-dimension N has boundary of dimension $< N$. Kontsevich-style integration into completion is possible if one takes $\Gamma = \mathbb{Z}$. The values $\sum a_n L^{-n}$ obtained as volumes are represented by rational functions.

Every formula has normal form $g(X^*)$ where $\widetilde{X} \subseteq \mathsf{VF}^{n+m} \times \Gamma^l$ is an $ACVF_A$- definable set, and g an $ACVF_A$- definable function on $\widetilde{X}$, and $X^* = \widetilde{X} \cap (\mathsf{VF}^n \times k^m) \times \Gamma^l$, such that:

(N) $g : X^* \to X$ is bijective.

Question: in char. 0, what about Cluckers-Loeser? Or H. - Kazhdan?

3) Further results of DKV:

a) Convolution.

b) GL_n.

4) Adeles. The Tamagawa number can be expressed motivically (vol$G(\mathcal{O})[G(\mathcal{O})\backslash G(\mathbb{A})_1/G(K)]$)

Motivic Weil; for $G = T$, Ono, Oesterlé.

5) Global integration: uniformity in Γ.

The contribution of logic to p-adic integration consists of uniformity in $\mathbf{k}$ and in Γ. The present global theory is uniform in $\mathbf{k}$ only, and it is known that the induced structure on Γ cannot be linear; the number of rational curve on a Calabi-Yau would fit into this framework (Givental, Kontsevich). Dimension growth?