Hilbert's fifth problem and applications

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Nonstandard Extensions

Start with a mathematical universe V containing all relevant mathematical objects, e.g.

- $\mathbb{N}$, $\mathbb{R}$, our topological group G;
- various cartesian products of the above sets;
- the elements of the above sets and the power sets of the above sets;

Then extend to a *nonstandard* mathematical universe V^*:

- To each basic set A, we have its nonstandard extension $A^* \supseteq A$;
- To every function $f : A \to B$ between basic sets, we have the nonstandard extension $f : A^* \to B^*$.

Transfer Principle

If S is a bounded first-order statement about objects in V, then it is true in V if and only if it is true in V^*.

The Transfer Principle in Action

Example

Let $(G, \cdot, 1)$ be a group. Then the following are true in V:

- $(\forall x \in G)(\exists y \in G)[(x \cdot y = 1) \text{ and } (y \cdot x = 1)]$
- $(\forall x \in G)(x \cdot 1 = 1 \cdot x = x)$
- $(\forall x \in G)(\forall y \in G)(\forall z \in G)[(x \cdot y) \cdot z = x \cdot (y \cdot z)]$.

By transfer, the following are true in V^*:

- $(\forall x \in G^*)(\exists y \in G^*)[(x \cdot y = 1) \text{ and } (y \cdot x = 1)]$
- $(\forall x \in G^*)(x \cdot 1 = 1 \cdot x = x)$
- $(\forall x \in G^*)(\forall y \in G^*)(\forall z \in G^*)[(x \cdot y) \cdot z = x \cdot (y \cdot z)]$.

In other words, $(G^*, \cdot, 1)$ is not only a *group, but it is an actual group.

Ultrafilters

Definition

Given a set I, a *filter* on I is a set $\mathcal{F} \subseteq \mathcal{P}(I)$ such that:

- $\emptyset \notin \mathcal{F}$
- $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$;
- $A \in \mathcal{F}$, $A \subseteq B \Rightarrow B \in \mathcal{F}$.

A maximal filter is called an *ultrafilter*; equivalently, a filter $\mathcal{U}$ is an ultrafilter if for all $A \in \mathcal{P}(I)$, either $A \in \mathcal{U}$ or $I \setminus A \in \mathcal{U}$.

An ultrafilter $\mathcal{U}$ is *principal* if there is $i \in I$ such that $\mathcal{U} = \{A \in \mathcal{P}(I) \ : \ i \in A\}$.

Ultrapowers

- Given a set X and a nonprincipal ultrafilter $\mathcal{U}$ on a set I, we set $X^{\mathcal{U}} := X^I / \sim_{\mathcal{U}}$, an *ultrapower* of X; here $f \sim_{\mathcal{U}} g$ if and only if $\{i \in I \ : \ f(i) = g(i)\} \in \mathcal{U}$.
- We identify X as a subset of $X^{\mathcal{U}}$ via the diagonal embedding $x \mapsto [(x, x, x, \ldots)]_{\mathcal{U}}$.
- Then $X^{\mathcal{U}}$ serves as a nonstandard extension of X: a function $f : X \to Y$ naturally extends to a function $f : X^{\mathcal{U}} \to Y^{\mathcal{U}}$ by $f([(x_i)]_{\mathcal{U}}) := [f(x_i)]_{\mathcal{U}}$.
- The Transfer Principle is more comonly known as the Łoś theorem.

Internal Sets

- For each basic set X, it is possible to identify $\mathcal{P}(X)^*$ with a subset of $\mathcal{P}(X^*)$. The elements of $\mathcal{P}(X)^*$ are called **internal** subsets of X^*. Subsets of X^* that are not internal are called **external.**
- It is the case that A^* for any A in V is internal, and any set defined from internal parameters in a first-order way is internal. (**Internal Definition Principle**)
- In the ultraproduct construction, internal subsets of X^* are sets of the form $\prod_{\mathcal{U}} B_i$, where $B_i \subseteq X$ for each i.

Saturation

The particularly useful nonstandard extensions are very rich in a certain technical sense.

Definition

Let κ be an infinite cardinal. We say that V^* is κ-**saturated** if whenever $(\mathcal{O}_i \mid i < \kappa)$ is a family of *internal* sets such that any intersection of a finite number of them is nonempty, then the intersection of all of them is nonempty.

We will assume our V^* is κ-saturated for a suitably large κ.

An Example of Saturation: Infinitesimals

- For $i \in \mathbb{N}$, let $\mathcal{O}_i := \{x \in \mathbb{R}^* \mid 0 < |x| < \frac{1}{i}\}$. By the internal definition principle, each $\mathcal{O}_i$ is internal.
- Clearly any finite intersection of the $\mathcal{O}_i$ is nonempty and so saturation yields that there is $x \in \bigcap_{i \in \mathbb{N}} \mathcal{O}_i$. Such an x is positive but smaller than any *standard* real number, i.e. x is an infinitesimal.
- Since $\mathbb{R}$ is an ordered field and the ordered field axioms are first-order, one has that $\mathbb{R}^*$ is an ordered field extension of $\mathbb{R}$. Hence, $\frac{1}{x}$ is an element of $\mathbb{R}^*$ bigger than any *standard* real number, i.e. $\frac{1}{x}$ is an *infinite* element of $\mathbb{R}$.
- In an ultraproduct with $I = \mathbb{N}$, then $[(1, 2, 3, \ldots)]_{\mathcal{U}} \in \mathbb{N}^*$ is infinite and $[(1, \frac{1}{2}, \frac{1}{3}, \ldots)]_{\mathcal{U}} \in \mathbb{R}^*$ is infinitesimal.

Infinitesimals in Hausdorff Spaces

- Suppose X is a Hausdorff topological space. For any $a \in X$, we set $\mu(a)$ to be the intersection of the sets $\mathcal{O}^*$, where $\mathcal{O}$ is a neighborhood of a. If a is not isolated, then by saturation we can find $a' \in \mu(a) \setminus \{a\}$; such an a' is *infinitely close to a*.
- We let $X_{ns} := \bigcup_{a \in X} \mu(a)$, the set of *nearstandard* elements of X^*. The Hausdorff axiom implies that $\mu(a) \cap \mu(b) = \emptyset$ if $a \neq b$. Thus, if $a' \in X_{ns}$, we can write $\mathrm{st}(a')$, read the *standard part* of a', for the unique element of X that a' is infinitely close to. If $a, b \in X_{ns}$ are such that $\mathrm{st}(a) = \mathrm{st}(b)$, we write $a \sim b$.
- Important: If a is not isolated in X, then $\mu(a)$ is external. To see this, suppose it were internal. Then the family $\{\mathcal{O}^* \setminus \mu(a) \mid \mathcal{O} \text{ a neighborhood of } a\}$ of nonempty internal sets would have the finite intersection property (since a is not isolated) yet would have empty intersection.

Continuity and convergence

As an example of the utility of nonstandard extensions, we have the following (intuitive) characterizations of continuity and convergence:

Lemma

Suppose X and Y are Hausdorff spaces, a is an element of X, and $f : X \to Y$ is a function. Then f is continuous at a if and only if for every $a' \in \mu(a)$, we have $f(a') \in \mu(f(a))$.

Lemma

If X is a Hausdorff space, then a sequence (a_n) from X converges if and only if $a_\sigma \sim a_\nu$ for all $\sigma, \nu \in \mathbb{N}^* \setminus \mathbb{N}$, in which case $\lim_{n\to\infty} a_n = \mathrm{st}(a_\sigma)$ for any $\sigma \in \mathbb{N}^* \setminus \mathbb{N}$.

Nonstandard topology

We can characterize the interior and closure of a set in the following useful nonstandard way:

Lemma

Suppose that X is a Hausdorff space, $F \subseteq X$, $a \in F$.

- *$a \in \mathrm{int}(F)$ if and only if $\mu(a) \subseteq F^*$.*
- *$a \in \overline{F}$ if and only if $\mu(a) \cap F^* \neq \emptyset$.*

In particular, if $b \in X_{\mathrm{ns}} \cap F^*$, then $\mathrm{st}(b) \in \overline{F}$.

Compactness also has a nice characterization:

Lemma

Suppose that X is a Hausdorff space. Then $F \subseteq X$ is compact if and only if $F^* \subseteq X_{\mathrm{ns}}$.

Hilbert's Fifth Problem

Definition

A topological group G is **locally euclidean** if there is a neighborhood of the identity homeomorphic to some $\mathbb{R}^n$.

Definition

G is a **Lie group** if G is a real analytic manifold which is also a group such that the maps $(x, y) \mapsto xy : G \times G \to G$ and $x \mapsto x^{-1} : G \to G$ are real analytic maps.

Hilbert's Fifth Problem (H5)

If G is a locally euclidean topological group, is there a real analytic structure on G compatible with the topology so that the group operations become real analytic?

Positive Answers to H5

- Linear Case: G can be continuously embedded into $\mathrm{Gl}_n(\mathbb{R})$ for some n (von Neumann)
- Abelian Case (Pontrjagin)
- Compact Case (Weyl)

Theorem (Gleason, Montgomery, Zippin- 1952)

For a locally compact (hausdorff) group G, the following are equivalent:

1. *G is locally euclidean;*
2. *G has **no small subgroups (NSS)**, i.e. there is a neighborhood of the identity containing no nontrivial subgroups of G;*
3. *G is a Lie group.*

- Nonstandard Exposition of the Full Solution: Hirschfeld (1990)

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A Nonstandard Exercise

From now on, μ will denote $\mu(1)$; we call μ the set of *infinitesimals* of G. Observe that μ is a normal subgroup of G^*.

Exercise

G is NSS if and only if μ has no internal subgroups other than $\{1\}$.

- The forward direction follows from the transfer principle.
- The backward direction follows from saturation.

A Model-Theoretic Application of H5

Theorem (Pillay-1988)

Suppose that G is a group definable in an o-minimal expansion of $(\mathbb{R},<)$ *and* $\dim_{\text{o-min}}(G) = n$*. Then there is a "definable" topology on G, called the t-topology, making G into a topological group, and a definable t-open set U containing the identity of G such that U is definably homeomorphic to a definable open subset of* $\mathbb{R}^n$ *(with the induced order topology). Consequently, G is an n-dimensional Lie group.*

One Parameter Subgroups

Definition

A **one-parameter subgroup of** G (1-ps of G) is a continuous group morphism $X : \mathbb{R} \to G$.

Put $L(G) := \{X : \mathbb{R} \to G \mid X \text{ is a 1-ps of } G\}$.

We have the **scalar multiplication** map $(r, X) \mapsto rX : \mathbb{R} \times L(G) \to L(G)$, where $(rX)(t) := X(rt)$.

We let O denote the trivial 1-ps of G, i.e. $O(t) \equiv 1$. Then:

- $0X = O$ and $1X = X$;
- $r(sX) = (rs)X$.

The Case of Lie Groups

- Suppose G is a Lie group and $X \in L(G)$. Then X is real analytic and so $X'(0) \in T_1(G)$. We get a bijection

$$X \mapsto X'(0) : L(G) \to T_1(G)$$

and the addition operation on $L(G)$ that makes this an $\mathbb{R}$-vector space isomorphism is given by

$$(X+Y)(t) = \lim_{n\to\infty} (X(\frac{1}{n})Y(\frac{1}{n}))^{[nt]}.$$

- Let $n = \dim G = \dim_{\mathbb{R}} L(G)$ and make $L(G)$ a real analytic manifold so that the linear isomorphisms $L(G) \cong \mathbb{R}^n$ are analytic isomorphisms.
- Then the **exponential map** $X \mapsto X(1) : L(G) \to G$ yields an analytic isomorphism from an open neighborhood of O in $L(G)$ onto an open neighborhood of 1 in G.

Plan of Proof for NSS implies Lie

Suppose G is locally compact and has NSS. One takes the following steps to prove that G is a Lie group.

1. Show that for any $X, Y \in L(G)$, there is an $X + Y \in L(G)$ given by

$$(X+Y)(t) = \lim_{n\to\infty} (X(\frac{1}{n})Y(\frac{1}{n}))^{[nt]}$$

   and that $L(G)$ becomes an $\mathbb{R}$-vector space under this addition and the aforementioned scalar multiplication. (In this talk, we will just sketch the proof of the existence of $X + Y$.)

2. Equip $L(G)$ with its compact-open topology and show that $L(G)$ becomes a topological vector space.

Plan of Proof for NSS implies Lie (cont'd)

3. Show that the exponential map

$$X \mapsto X(1) : L(G) \to G$$

   maps an open neighborhood of O in $L(G)$ onto an open neighborhood of 1 in G. Then since G is locally compact, so is $L(G)$, whence we conclude that $\dim_{\mathbb{R}}(L(G)) < \infty$. (This also shows that G is locally euclidean.)
4. Use the adjoint representation of G on $L(G)$ to conclude that G is a Lie group.

Adjoint Representation

- Let $\mathrm{Ad} : G \to \mathrm{Aut}(L(G))$ be defined by $\mathrm{Ad}(g)(X) = gXg^{-1}$, where $(gXg^{-1})(t) := gX(t)g^{-1}$. Then Ad is a continuous group morphism.
- Since $G/\ker(\mathrm{Ad})$ continuously embeds into $\mathrm{Aut}(L(G)) \cong \mathrm{Gl}_n(\mathbb{R})$, where $n := \dim_{\mathbb{R}}(L(G))$, von Neumann's Theorem implies that $G/\ker(\mathrm{Ad})$ is a Lie group.
- If G is connected (which we may suppose it is), then $\ker(\mathrm{Ad}) = \mathrm{center}(G)$, which is abelian, and hence a Lie group by Pontrjagin.
- Now use a result of Kuranishi which says that if N and G/N are Lie groups and there exists a *local cross-section* $G/N \rightharpoonup G$ (which will exist since G is NSS), then G is also a Lie group.

Powers of Infinitesimals

Since μ is a group, we have that $a^n \in \mu$ for any $a \in \mu$ and $n \in \mathbb{N}$. What about infinite powers of a?

Internal Induction

If $A \subseteq \mathbb{N}^*$ is *internal*, contains 0 and is closed under the successor operation, then $A = \mathbb{N}^*$.

Let $a \in \mu$ and let $A = \{\sigma \in \mathbb{N}^* \mid a^\sigma \in \mu\}$. Then A contains 0 and is closed under successor by continuity of multiplication. The problem is that A is not internal since μ is an *external* notion. Hence we cannot conclude that $a^\sigma \in \mu$ for all $\sigma \in \mathbb{N}^*$. In fact, if G has NSS, then $a^\sigma \notin \mu$ for some $\sigma \in \mathbb{N}^* \setminus \mathbb{N}$.

Landau Notation

We will need to use the following notation:

Notation

Suppose $i \in \mathbb{Z}^*$ and $\sigma \in \mathbb{N}^* \setminus \{0\}$.

- Say $i = o(\sigma)$ if $|i| < \frac{\sigma}{n}$ for every $n \in \mathbb{N}$;
- Say $i = O(\sigma)$ if $|i| < n\sigma$ for some $n \in \mathbb{N}$.

Intuitively, $i = o(\sigma)$ means that $\frac{|i|}{\sigma}$ is infinitesimal (so $|i|$ is in a strictly smaller archimedean class than σ) and $i = O(\sigma)$ means that $\frac{|i|}{\sigma}$ is an element of $\mathbb{R}_{ns}$ (so $|i|$ is not in archimedean class strictly greater than σ.)

Infinitesimal generators of 1-parameter subgroups

Fix $\sigma \in \mathbb{N}^* \setminus \mathbb{N}$.

Definition

Let $G(\sigma) := \{a \in \mu \mid a^i \in \mu \text{ for all } i = o(\sigma)\}$.

Lemma

If $a \in G(\sigma)$, then $a^i \in G_{\text{ns}}$ for all $i = O(\sigma)$.

Proof.

Fix a compact neighborhood U of 1. If $a^i \in U^*$ for all $i = O(\sigma)$, we are done. Otherwise, let $j \in \mathbb{N}^*$ be minimal such that $a^j \notin U^*$. Observe that $\sigma = O(j)$ and $a^j \in G_{\text{ns}}$. For $i = O(\sigma)$, write $|i| = nj + m$ with $m < j$ and $n \in \mathbb{N}$. Then $a^{|i|} = (a^j)^n \cdot a^m \in G_{\text{ns}}$ (since G_{ns} is a subgroup of G^*). □

Infinitesimal generators of 1-parameter subgroups

Lemma

For $a \in G(\sigma)$, define $X_a : \mathbb{R} \to G$ by $X_a(t) := \mathrm{st}(a^{[t\sigma]})$. Then X_a is a 1-ps of G.

Proof.

- It is easy to see that X_a is a homomorphism.
- To see that X_a is continuous, it suffices to check continuity at 0. Fix U a neighborhood of 1 in G and let V be a neighborhood of 1 in G such that $\overline{V} \subseteq U$.
- Let $A = \{n \in \mathbb{N}^* \mid a^k \in V^* \text{ for all } k \in \mathbb{Z}^* \text{ with } |k| < \frac{\sigma}{n}\}$. Then $A \subseteq \mathbb{N}^*$ is internal and contains all elements of $\mathbb{N}^* \setminus \mathbb{N}$ by assumption.
- Thus, by "underflow," there is $n \in \mathbb{N} \cap A$. Consequently, $X_a(t) = \mathrm{st}(a^{[t\sigma]}) \in \overline{V} \subseteq U$ for $t \in (-\frac{1}{n}, \frac{1}{n})$.

Infinitesimal generators of 1-parameter subgroups of G

Question

Which elements of $L(G)$ are of the form X_a for some $a \in G(\sigma)$?

Answer

All of them! Suppose $X \in L(G)$ and let $a := X(\frac{1}{\sigma}) \in \mu$. Then if $i = o(\sigma)$, we have $a^i = X(\frac{i}{\sigma}) \in \mu$, whence $a \in G(\sigma)$, and

$$X_a(t) = \text{st}(a^{[t\sigma]})) = \text{st}(X(\frac{[t\sigma]}{\sigma})) = X(t).$$

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Infinitesimal Generators of 1-parameter subgroups of G

One can ask the question "When does $X_a = O$?" This happens if and only if $a^i \in \mu$ for all $i = \mathrm{O}(\sigma)$. Those infinitesimals are important enough to merit a definition.

Definition

$G^o(\sigma) := \{a \in \mu \mid a^i \in \mu \text{ for all } i = \mathrm{O}(\sigma)\}$.

So we have $X_a = O$ if and only if $a \in G^o(\sigma)$ and observe that $1 \in G^o(\sigma)$, $G^o(\sigma) \subseteq G(\sigma)$ and $G^o(\sigma)$ is closed under inverses.

Gleason-Yamabe Lemmas

One next wants to show that $G(\sigma)$ is a subgroup of μ with normal subgroup $G^{o}(\sigma)$ and use this to help us put a group law on $L(G)$. To do this and more, we will need to know more about the growth rates of powers of elements of these sets. The ingenious idea of Hirschfeld was to translate the very technical lemmas of Gleason and Yamabe into clear and concise statements about such growth rates.

Lemma

- *Let $a_1, \ldots, a_\sigma$ be an internal sequence in G^* such that all $a_i \in G^{o}(\sigma)$. Then $a_1 \cdots a_\sigma \in \mu$;*
- *If $a \in G(\sigma)$ and $b \in G^{o}(\sigma)$, then $(ab)^i \sim a^i$ for all $i \leq \sigma$;*
- *Suppose $a, b \in G(\sigma)$ are such that $a^i \sim b^i$ for all $i \leq \nu$. Then $a^{-1}b \in G^{o}(\sigma)$.*

Consequences of the Gleason-Yamabe Lemmas

The following theorem follows rather easily from the aforementioned consequences of the Gleason-Yamabe Lemmas.

Theorem

- *$G(\sigma)$ is a group and $G^o(\sigma)$ is a normal subgroup;*
- *The quotient group $G(\sigma)/G^o(\sigma)$ is abelian;*
- *For $a, b \in G(\sigma)$, $X_a = X_b$ if and only if $a^{-1}b \in G^o(\sigma)$;*
- *The surjective map $a \mapsto X_a : G(\sigma) \to G^o(\sigma)$ induces a bijection $aG^o(\sigma) \mapsto X_a : G(\sigma)/G^o(\sigma) \to L(G)$.*

Group Law on $L(G)$

We make $L(G)$ into an abelian group with the operation $+_\sigma$ so that the aforementioned bijection is an abelian group isomorphism. More specifically, $X_a +_\sigma X_b := X_{ab}$, or in other words, $(X +_\sigma Y)(t) = \mathrm{st}((X(\frac{1}{\sigma})Y(\frac{1}{\sigma}))^{[\sigma t]})$. Using the Gleason-Yamabe lemmas again, one can prove the following:

Fact

If $\sigma, \nu \in \mathbb{N}^* \setminus \mathbb{N}$, then $X +_\sigma Y = X +_\nu Y$ for all $X, Y \in L(G)$.

Corollary

$X + Y$ exists in $L(G)$, i.e. for every $t \in \mathbb{R}$, we have $\lim_{n\to\infty}(X(\frac{1}{n})Y(\frac{1}{n}))^{[nt]}$ exists.

Yamabe's Theorem

Theorem (Yamabe)

If G is a locally compact group, then there is an open subgroup G' of G such that G' is a generalized Lie group, that is, G' is the projective limit of a sequence of Lie groups. More precisely, for every neighborhood U of the identity in G', there is a compact normal subgroup H of G' such that G'/H is a Lie group.

The proof actually shows that there are arbitrarily small H such that G'/H is NSS. It then follows from the H5 that G'/H is a Lie group.

Yamabe's theorem is crucial in Hrushovski's work on approximate subgroups.

An Application: Gromov's Theorem

Theorem (Gromov)

Let G be a finitely generated group of polynomial growth. Then G is virtually nilpotent.

- Let X be the asymptotic cone of the Cayley graph of G. Then $L := \text{Isom}(X)$ is locally compact with its compact-open topology.
- One can use Yamabe (and other facts) to show that L is a Lie group with finitely many connected components.
- Using that L is a Lie group, namely NSS and that $L/\ker(\text{Ad})$ embeds into a linear group (so one can use Jordan's theorem on finite subgroups of Gl_n and the Tits alternative), one gets a surjective homomorphism $H \to \mathbb{Z}$, where $[G : H] < \infty$. This is the key step in the proof of the theorem.

What is a local group?

Definition

A **local group** is a tuple $(G, 1, \iota, p)$ where:

- G is a hausdorff topological space with distinguished element $1 \in G$;
- $\iota : \Lambda \to G$ is continuous, where $\Lambda \subseteq G$ is open;
- $p : \Omega \to G$ is continuous, where $\Omega \subseteq G \times G$ is open;
- $1 \in \Lambda$, $\{1\} \times G \subseteq \Omega$, $G \times \{1\} \subseteq \Omega$;
- $p(1, x) = p(x, 1) = x$;
- if $x \in \Lambda$, then $(x, \iota(x)) \in \Omega$, $(\iota(x), x) \in \Omega$, and

$$p(x, \iota(x)) = p(\iota(x), x) = 1;$$

- if $(x, y), (y, z) \in \Omega$ and $(p(x, y), z), (x, p(y, z)) \in \Omega$, then

$$p(p(x, y), z) = p(x, p(y, z)).$$

The Local H5

The Local H5

Is every locally euclidean *local group* locally isomorphic to a Lie group?

- Jacoby gave a “proof” of a positive answer to the local H5 in 1957.
- Jacoby’s proof was invalidated by Plaut in the early ’90s.
- We adapt the nonstandard methods employed by Hirschfeld to give a proof of a positive answer to the Local H5.
- Salvaging the Local H5 was a necessary endeavor. For example, the solution of Hilbert’s Fifth Problem for cancellative semigroups on manifolds rested on the truth of the Local H5.

A Simple Example of a Local Group

Let $G = (-1, 1)$. Then G is a local group under addition, where we take $\Lambda = G$ and $\Omega = \{(x, y) \in G \mid x + y \in G\}$.

More generally, if H is a topological group and U is an open neighborhood of the identity, then we obtain a local group $G := U$, where the local group operations are those inherited from H, $\Lambda = \{x \in G \mid x^{-1} \in G\}$ and $\Omega = \{(x, y) \in G \times G \mid xy \in G\}$.

These are very special types of local groups, namely the *globalizable local groups*.

Further Notions Associated to Local Groups

Definition

Suppose G is a local group.

1. G is **locally euclidean** if there is an open neighborhood of 1 in G homeomorphic to some $\mathbb{R}^n$;
2. G is a **local Lie group** if G admits a real analytic structure such that the maps ι and p are real analytic;
3. Let U be an open neighborhood of 1 in G. Then the **restriction of G to U** is the local group $G|U := (U, 1, \iota|\Lambda_U, p|\Omega_U)$, where
$$\Lambda_U := \Lambda \cap U \cap \iota^{-1}(U) \text{ and } \Omega_U := \Omega \cap (U \times U) \cap p^{-1}(U).$$
4. G is **globalizable** if there is a topological group H and an open neighborhood U of 1_H in H such that $G = H|U$.

Not every local group is globalizable

Consider the local Lie group given by $G = \mathbb{R}$,

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid |xy| \neq 1\}, \quad \Lambda = G \setminus \{\frac{1}{2}, 1\},$$

with multiplication and inversion

$$p(x, y) = \frac{2xy - x - y}{xy - 1}, \iota(x) = \frac{x}{2x - 1}.$$

(Note that 0 is the neutral element.) Then G is not globalizable since $p(x, 1) = p(1, x) = 1$ for all $x \neq \pm 1$.

Let $U = \{|x| < \frac{1}{2}\}$. Then $x \mapsto \frac{x}{x-1} : U \to \mathbb{R}$ shows that $G|U$ is globalizable.

The Local H5-Two Forms

Local H5-First Form

If G is a locally euclidean local group, then some restriction of G is a local Lie group.

Local H5-Second Form

If G is a locally euclidean local group, then some restriction of G is globalizable. (In short: Is G locally isomorphic to a group?)

The equivalence of the two forms follows from the positive solution to the original H5 and the fact (due to Cartan) that every local Lie group has a restriction which is the restriction of a global Lie group.

What was wrong with Jacoby's Proof?

- We say that G is **globally associative** if, given any finite sequence of elements from G, if there are two ways of introducing parentheses such that both products thus formed exist, then the two products are in fact equal.
- Jacoby's Theorem 8 essentially claims that every local group is globally associative.
- This is not true in general. In fact, it is a theorem of Mal'cev that a local group is globally associative if and only if it is globalizable.
- Olver constructs local Lie groups which are associative up to sequences of length n for a given n but which are not associative for sequences of length $n+1$.

Associativity

Suppose a, b, c, d reside in a local group G and $a(b(cd))$ and $((ab)c)d)$ are both defined. Why aren't they necessarily equal, as Jacoby thought they were?

The usual calculation:

$$a(b(cd)) = (ab)(cd) = ((ab)c)d.$$

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$$a(b(cd)) = (ab)(cd) = ((ab)c)d.$$

Problem: $(ab)(cd)$ may not be defined!

Jacoby's Paper

Why did it take so long for the flaw in Jacoby's proof to be discovered?

- His paper was written in Quine's New Foundations.
- Of the 106 propositions in his paper, more than half say "Proof Omitted."
- "A proof is omitted and no reference is given when it is believed that a persion familiar with the classical theory of topological groups can readily supply the proof. Frequently, the main difficulty is in finding a suitable statement of the local version of a global theorem; once the proper form is found the alteration of the proof is trivial. A proof is omitted and a reference is given in case the global version cannot yet be considered as classical."

Theorem 8

If $\mathfrak{G}$ is a local group, if $n, m \in \mathbb{N}$ and if $O \in \psi\mathfrak{G}$, then (i) for all $A \in \Phi(\mathfrak{G}, n+m)$, $A \in \Phi(\mathfrak{G}, n) \cap \Phi(\mathfrak{G}, m)$ and $A^m \cdot A^n = A^{m+n}$; (ii) for all

$$A \in \Phi(\mathfrak{G}, O, n) \cap \Phi(\mathfrak{G}, O, m)$$

so that $A \subseteq A \cdot A$ and $A^m \cdot A^n \subseteq O$, $A \in \Phi(\mathfrak{G}, O, n+m)$; (iii) for all

$$A \in \Phi(\mathfrak{G}, O, n), \quad A' \in \Phi(\mathfrak{G}, O, n)$$

and $(A')^n = [A^n]'$; (iv) for all $A \in \Phi(\mathfrak{G}, B, n)$ and $B \in \Phi(\mathfrak{G}, O, m)$,

$$A \in \Phi(\mathfrak{G}, O, mn)$$

and $A^{mn} \subseteq B^m$; (v) for all $A \in \Phi(\mathfrak{G}, O, mn)$, $A \in \Phi(\mathfrak{G}, O, m)$, $A^m \in \Phi(\mathfrak{G}, O, n)$ and $(A^m)^n = A^{mn}$.

Proof Omitted.

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Proof Omitted.

Another example:

Theorem 9

If $\mathfrak{G}$ is a local group, if $O \in \Psi\mathfrak{G}$ and if $\{a\}$, A, $\{a\} \cdot A \subseteq O$, then $[x : A : a \cdot x]$ maps $A\#\mathfrak{D}$ homeomorphically onto $(\{a\} \cdot A)\#\mathfrak{D}$; the inverse is

$$[x : \{a\} \cdot A : a' \cdot x].$$

Similarly, if $\{b\}$, B, $B \cdot \{b\} \subseteq O$, then $[x : B : x \cdot b]$ maps $B\#\mathfrak{D}$ homeomorphically onto $(B \cdot \{b\})\#\mathfrak{D}$; the inverse is $[x : B \cdot \{b\} : x \cdot b']$. Finally, if $C \subseteq O$, then $[x : C : x']$ maps $C\#\mathfrak{D}$ homeomorphically onto $C'\#\mathfrak{D}$; the inverse is $[x : C' : x']$.

Proof omitted.

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If $\mathfrak{G}$ is a local group, if $O \in \Psi\mathfrak{G}$ and if $\{a\}, A, \{a\} \cdot A \subseteq O$, then $[x : A : a \cdot x]$ maps $A\#\mathfrak{D}$ homeomorphically onto $(\{a\} \cdot A)\#\mathfrak{D}$; the inverse is

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Similarly, if $\{b\}, B, B \cdot \{b\} \subseteq O$, then $[x : B : x \cdot b]$ maps $B\#\mathfrak{D}$ homeomorphically onto $(B \cdot \{b\})\#\mathfrak{D}$; the inverse is $[x : B \cdot \{b\} : x \cdot b']$. Finally, if $C \subseteq O$, then $[x : C : x']$ maps $C\#\mathfrak{D}$ homeomorphically onto $C'\#\mathfrak{D}$; the inverse is $[x : C' : x']$.

Proof omitted.

A Way Around Global Associativity

It will be the case that we can "unambiguously multiply" sequences of elements of any length, provided the elements are close enough to the identity, as we now explain.

Definition

Let $a_1, \ldots, a_n, b \in G$ with $n \geq 1$. We define the notion $(a_1, \ldots, a_n) \to b$ by induction on n as follows:

- $(a_1) \to b$ iff $a_1 = b$;
- $(a_1, \ldots, a_{n+1}) \to b$ iff for every $i \in \{1, \ldots, n\}$, there exists $b_i', b_i'' \in G$ such that $(a_1, \ldots, a_i) \to b_i'$, $(a_{i+1}, \ldots, a_{n+1}) \to b_i''$, $(b_i', b_i'') \in \Omega$, and $b_i' \cdot b_i'' = b$.

We say that $a_1 \cdots a_n$ is **defined** if there exists a (necessarily unique) $b \in G$ such that $(a_1 \ldots, a_n) \to b$. If $a_i = a$ for each i, we say that a^n is defined.

The Sets $\mathcal{U}_n$

Terminology

- If $W \subseteq \Lambda$ and $\iota(W) \subseteq W$, we say that W is **symmetric.**
- For a set A, $A^{\times n} := \underbrace{A \times \cdots \times A}_{n \text{ times}}$;

Lemma

There are open symmetric neighborhood $\mathcal{U}_n$ of 1 for $n > 0$ such that $\mathcal{U}_{n+1} \subseteq \mathcal{U}_n$ and for all $(a_1, \ldots, a_n) \in \mathcal{U}_n^{\times n}$, $a_1 \cdots a_n$ is defined.

G^*

Set $\mu := \mu(1) \subseteq G^*$. Note that $\mu \times \mu \subseteq \Omega^*$ and that μ **is a group with the induced multiplication from** G^*. Here are some important observations that allow one to adapt Hirschfeld's methods to the local group setting:

- If $(a, b) \in \Omega$ and $a' \in \mu(a)$ and $b' \in \mu(b)$, then $(a', b') \in \Omega^*$ and $a'b' \in \mu(ab)$.
- If $a \in G^*_{ns}$ and $b \in \mu$, then $(a, b), (b, a) \in \Omega^*$ and $ab, ba \sim a$.
- For any $a \in G$ and $a' \in \mu(a)$, if a^n is defined for some $n \in \mathbb{N}$, then $(a')^n$ is defined and $(a')^n \in \mu(a^n)$. In particular, the partial function $p_n : G \rightharpoonup G$ given by $p_n(a) = a^n$, if a^n is defined, has an open domain and is continuous.

A Central Question

Bad Question

Which local groups are globalizable?

Better Question

Which local groups are "locally" globalizable? In other words: Which local groups are locally isomorphic to groups?

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Locally Compact Local Groups

Theorem (van den Dries, G.)

If G is a locally compact local group, then G is locally isomorphic to a group.

Remarks

1. By Yamabe's theorem, it follows that every locally compact local group is locally isomorphic to a generalized Lie group.
2. The above theorem was proven for local Lie groups by Cartan and holds for locally euclidean local groups as a corollary of the affirmative solution to the local H5.
3. The above theorem is optimal in the sense that the locally compact assumption cannot be removed. Indeed, there are local Banach-Lie groups which are not locally isomorphic to groups.

Ingredient #1: A Theorem of Swierczkowski

In the proof of the above theorem, we use the following result, which is a special case of a more general result do to Swierczkowski.

Fact

If there is a surjective *strong* morphism $G \to L|V$, where L is a Lie group and V is a symmetric open neighborhood of 1 in L, then G is locally isomorphic to a topological group.

(A morphism $f : G \to H$ is *strong* if, for all $a, b \in G$, we have $(a, b) \in \Omega_G$ if and only if $(f(a), f(b)) \in \Omega_H$, in which case $f(ab) = f(a)f(b)$.)

Proof Idea

Use “local group cohomology” to identify the obstruction to globalizability. In the setting of our fact, this cohomology can be reformulated into a statement about $H_2(L)$ and $H_2(L) = 0$ for a simply connected Lie group.

Ingredient #2: A Local Yamabe

Theorem (G.)

If G is locally compact, then there is a restriction $G|U$ of G and a compact normal subgroup N of $G|U$ such that $(G|U)/N$ has NSS.

- By the above theorem, given locally compact G, we can assume, without loss of generality, that there is a compact subgroup N of G such that G/N is NSS.
- By the solution to the local H5, we have that G/N is locally isomorphic to a Lie group L.
- Thus the surjective strong morphism $G \to G/N$ restricts to a surjective strong morphism $G|U \to L|V$ for appropriate U and V.
- Thus, by Swierczkowski, we have that $G|U$ (and hence G) is locally isomorphic to a group.

A Key Nonstandard Ingredient

Lemma

Suppose V is a neighborhood of 1 in G. Then V contains a compact subgroup H of G and a neighborhood W of 1 in G such that every subgroup of G contained in W is contained in H.

- To see this, fix an internally open neighborhood W of 1 contained in μ.
- By the local Gleason-Yamabe Lemmas (see next slide), for every internal sequence $E_1, \ldots, E_\nu$ of internal subgroups of G^* contained in W, we have that $E_1 \cdots E_\nu$ is defined and contained in μ.
- Let S be the union of all such $E_1 \cdots E_\nu$. Then S is an internal subgroup of G^* contained in μ and every internal subgroup of G^* contained in W is contained in S.
- Let H be the internal closure of S in G^*. Use transfer with this H and W.

A Nonstandard Gleason-Yamabe Lemma

Lemma

Suppose $\nu > \mathbb{N}$ *and* $a_1, \ldots, a_\nu \in \mu$ *is an internal sequence such that* $a_i \in G^o(\nu)$ *for all* $i \in \{1, \ldots, \nu\}$*. Then* $a_1 \cdots a_\nu$ *is defined and* $a_1 \cdots a_\nu \in \mu$*.*

The proof that $a_1 \cdots a_\nu \in \mu$ if $a_1 \cdots a_\nu$ is defined mimics the proof of the global version of the lemma and is a consequence of the Gleason-Yamabe Lemmas. We will now indicate how to prove that $a_1 \cdots a_\nu$ is defined.

Fix an internal $E \subseteq \mu$. We will prove the following:

Claim

If $b_1, \ldots, b_\nu$ is an internal sequence such that b_i^j is defined and $b_i^j \in E$ for all $i, j \in \{1, \ldots, \nu\}$, then $b_1 \cdots b_\nu$ is defined.

A Nonstandard Gleason-Yamabe Lemma (cont'd)

Claim

If $b_1, \ldots, b_\nu$ is an internal sequence such that b_i^j is defined and $b_i^j \in E$ for all $i, j \in \{1, \ldots, \nu\}$, then $b_1 \cdots b_\nu$ is defined.

We prove the claim by internal induction on ν. At the induction step, the induction hypothesis allows us to assume that $b_1 \cdots b_\nu$ is defined and $b_i \cdots b_{\nu+1}$ is defined for every $i \in \{2, \cdots, \nu+1\}$. By the first part of the proof of the Lemma, we can conclude, for every $i \in \{1, \ldots, \nu\}$

$$(b_1 \cdots b_i, b_{i+1} \cdots b_{\nu+1}) \in \mu \times \mu \subseteq \Omega^*,$$

which by the transfer of the associativity condition is enough to conclude that $b_1 \cdots b_{\nu+1}$ is defined.

The claim finishes the proof of the theorem by taking E to be the internal set of all a_i^j where $i, j \in \{1, \ldots, \nu\}$.

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