Doubling coverings and Doubling inequalities

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Contents

Doubling coverings

Doubling coverings (doubling parametrizations)

- We replace the covering sets by the images of the unit ball $B_1 \subset \mathbb{C}^n$ by a certain set of functions.

Definition

A doubling covering $\mathcal{U}$ of a relatively compact domain G in a complex n-dimensional manifold Y consists of a finite collection of analytic univalent mappings $\psi_j : B_1 \to Y$ so that

1) Their images (aka charts) $U_j = \psi_j(B_1)$ cover G.

2) Each ψ_j is extendable to a univalent mapping $\tilde{\psi}_j : B_4 \to Y$.

- We denote by $\kappa(\mathcal{U})$ the minimal number of images in $\mathcal{U}$.
- Of course, B_4 can be replaced by B_γ for any $\gamma > 1$.

Doubling chains

- We call two charts U_i and U_j in $\mathcal{U}$ neighboring, if $U_i \cap U_j \neq \emptyset$.

Definition

A chain Ch in a covering $\mathcal{U}$ is a sequence $\{j_1, j_2, \ldots, j_n\}$ of pairwise different indices, such that $U_{j_p}, U_{j_{p+1}}$ are neighboring for each $p = 1, \ldots, n-1$.

- For two neighboring charts U_i and U_j we define their intersection radius as the maximal radius $\rho > 0$ such that both $\psi_i^{-1}(U_i \cap U_j) \subset B_1 \subset \mathbb{C}^n$, and $\psi_j^{-1}(U_i \cap U_j) \subset B_1 \subset \mathbb{C}^n$ contain sub-balls of radius ρ (not necessarily concentric with B_1).
- We denote by $\rho(\mathcal{U})$ the minimum which is taken over all intersections radii of all the neighboring charts.

Some remarks

- Doubling coverings provide, essentially, a conformally invariant version of the Whitney's balls coverings of a domain $W \subset \mathbb{R}^n$.
- These coverings consist of balls B_j such that larger concentric balls γB_j are still in W.
- So, in our definition we replace W by a complex manifold Y, while the balls B_j are replaced by the charts U_j.
- The assumption of univalence of ψ_j can be, presumably, omitted in most of applications.
- You should think about Y as a level hypersurface of a polynomial on $\mathbb{C}^n$ and G as its intersection with the unit cube Q.

Motivation & History

- The introduction and study of doubling coverings is motivated mostly by the fact that they form a special class of "smooth parameterizations", which are used in smooth dynamics, diophantine geometry, etc.
- There are prominent open problems in dynamics and in diophantine geometry, where constructing certain parameterizations, and bounding their complexity, are expected to be important.
- In fact, a doubling covering is a complex counterpart of a "real analytic parameterization", as introduced and developed by Yomdin '91. There are other types of "smooth parameterizations": C^k, mild, etc.
- In recent years a lot of progress has been done: Binyamini, Novikov, Burguet, Liao, Yang, Cluckers, Pila, Wilkie, etc.

Kobayashi metric

- The Kobayashi metric is an intrinsic pseudo-metric associated to a complex manifold.
- The Kobayashi distance $d(p, q)$ is defined as follows:
  - Choose points $p = p_0, p_1, \dots, p_{k-1}, p_k = q \in Y$, points $a_1, \dots, a_k$, $b_1, \dots, b_k$ in the unit disk $D_1 \subset \mathbb{C}$, and holomorphic mappings $f_1, \dots, f_k$ from D_1 to Y, such that $f_i(a_i) = p_{i-1}$, $f_i(b_i) = p_i$, $i = 1, \dots, k$.
  - Form a sum $\sum_{i=1}^{k} \rho(a_i, b_i)$, where ρ is a Poincaré metric on D_1, and put $d(p, q)$ to be the infimum of these sums for all possible choices.
- The complexity of doubling coverings of Y depends only on the complex analytic structure of Y, and so one can hope to define, in its terms, certain invariants of Y.
- We make an initial step in this direction, showing that the complexity of a doubling coverings $\mathcal{U}$, bounds the Kobayashi distance on Y.

Doubling coverings and Kobayashi metric

- Let Y be a complex n-dimensional manifold.
- Let $G \subset Y$ be a connected relatively compact domain and $p, q \in G$.
- Let $\mathcal{U}$ be a 4-doubling covering of G in Y.
- Let Ch be a chain in $\mathcal{U}$ joining p, q.

Corollary

The Kobayashi distance satisfies

$$d(p,q) \leq 3\ell(Ch) \leq 3\kappa(\mathcal{U}).$$

Proof

- Let $U_1, \ldots, U_\ell$ be the charts in Ch, denote $p_0 = p, p_\ell = q$, and for $i = 1, \ldots, \ell - 1$ pick $p_i \in U_i \cap U_{i+1}$.
- Next, put $\tilde{a}_i = \psi_i^{-1}(p_{i-1})$, and $\tilde{b}_i = \psi_i^{-1}(p_i)$ in B_1.
- Now, define an affine map $T_i : D_1 \to B_4$, requiring the image $T_i(D_1)$ be the intersection disk $\tilde{D}$ of B_4 and of the complex line, passing through the points $\tilde{a}_i, \tilde{b}_i \in B_1$.
- Clearly, the radius of $\tilde{D}$ is at least 3, while the points $\tilde{a}_i, \tilde{b}_i \in B_1$ belong to a concentric subdisk of $\tilde{D}$ of radius at most 1.
- Finally, we put $a_i = T_i^{-1}(\tilde{a}_i)$, and $b_i = T_i^{-1}(\tilde{b}_i)$, and take $f_i = \psi_i \circ T_i$.
- It remains to notice that for each i our points a_i, b_i belong to the concentric disk $D_{1/3}$ of radius $\frac{1}{3}$ in D_1, and hence $\rho(a_i, b_i) \leq 3/2$.
- Indeed, the Poincaré metric on D_1 is given by $ds = \frac{2|dz|}{1-|z|^2}$. So inside $D_{1/3}$ we have $ds \leq 9/4|dz|$, and therefore the Poincaré distance $\rho(a_i, b_i)$ does not exceed $3/2$.

Whitney's balls coverings

Doubling *balls* coverings of a punctured cube

- Let $\Sigma = \{z_1, \dots, z_m\} \subset Q \subset \mathbb{R}^n$.
- Let Σ_δ be a δ-neighborhood of Σ.
- Consider the domain $Q_\delta = Q \setminus \Sigma_\delta$.
- That is, we removed from Q balls of radius $\delta > 0$ around each point $z_1, \dots, z_d$.

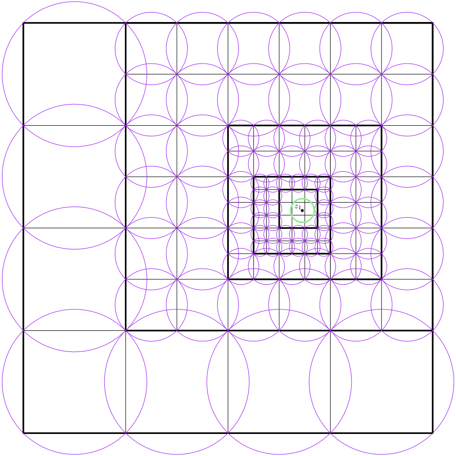
Definition

A γ-doubling balls covering $\mathcal{U}$ of Q_δ in $\mathbb{R}^n \setminus \{z_1, \dots, z_m\}$ is a collection of balls $B_j \subset \mathbb{R}^n$, which covers Q_δ such that the concentric balls γB_j are contained in $\mathbb{R}^n \setminus \Sigma_\delta$.

- Our construction is inspired by the classical Whitney's balls coverings. Similar constructions appear also in Calderón-Zygmind decomposition.

An example of a 2-dimensional cube with a point z_1

- Covering of a cube "avoiding" a δ-neighbourhood of a point z_1 with $\gamma = 2$.



- The balls B are the circumscribed balls of certain sub-cubes in the subdivisions of Q s.t. for any $B \in \mathcal{U}$ we have $\{z_1, \ldots, z_m\} \cap \gamma B = \emptyset$.

γ-doubling balls covering of a punctured cube

- In this special example, we provide an explicit (non-asymptotic) bound, depending on the number of the removed points, but not on their position.

Theorem

Let $\gamma > 1$. There is a γ-doubling balls covering $\mathcal{U}$ of Q_δ in $\mathbb{R}^n \setminus \Sigma$ so that

$$\kappa(\mathcal{U}) \leq m \cdot c \log(C/\delta).$$

where $c = (3\sqrt{n}\gamma)^n$, $C = 3n\gamma$.

- Moreover, for any v, w belonging to the same connected component of Q_δ, there exists a chain Ch in $\mathcal{U}$, joining v, w, such that for any two consecutive balls $B_{j_1}, B_{j_2} \in Ch$ the ratio of the radii of these balls is either $\frac{1}{2}, 1$ or 2, and the intersection $B_{j_1} \cap B_{j_2}$ contains a ball of the radius at least $1/3$ of the smaller of the radii.

Sharpness of this bound

- The bound is sharp with respect to the parameters m, γ and δ, up to coefficients depending only on the dimension n.

Example

Consider the case of only one point $z_1 = 0 \in \mathbb{R}^n$, and let $\mathcal{U}$ be a γ-doubling balls covering of Q_δ in $\mathbb{R}^n \setminus \{0\}$.

- Each ball $B(z_0, r) \in \mathcal{U}$, centered at z_0 of radius r, satisfies $\|z_0\| > \gamma r$.
- So to cover a spherical shell $\gamma r \leq \|z\| \leq \gamma r + 1$ we need at least $c_n \gamma^{n-1}$ balls in $\mathcal{U}$.
- Now, to "reach" the δ-neighborhood of 0 we need $\log(C_n \gamma/\delta)$ concentric spherical shells as above.
- Finally, for several points $z_1, \dots, z_m$, and for δ small enough, we can apply the above considerations to each point z_j separately.
- Altogether we obtain a lower bound for $\kappa(\mathcal{U})$ of the form

$$m \cdot c_n \gamma^{n-1} \log(C_n \gamma/\delta).$$

Bounds on the complexity

Generic polynomials

- Let P be a generic polynomial of degree $d \geq 2$ on $\mathbb{C}^n$.
- The singular set of P, $\Sigma_P = \{w_1, \dots, w_m\}$, consists of only isolated and non-degenerate critical points, and by Bézout theorem $m \leq (d-1)^n$.
- This assumption implies, for a normalized P, implies that for $z \in Q$

$$K_P \cdot \mathrm{dist}(z, \Sigma) \leq \|\nabla P(z)\| \leq nd^4 \mathrm{dist}(z, \Sigma)$$

  where $K_P > 0$ is a constant depending on P.
- The upper bound for $\|\nabla P(z)\|$ follows from Markov's inequality.
- However, the lower bound is valid only under the "general position" assumption. Let us stress that the constant K_P depends not only on the degree of P, but *on its specific coefficients*.
- Notice that we always have $K_P \leq nd^4$.

Settings & Problem

- Let P be a generic polynomial of degree $d \geq 2$ on $\mathbb{C}^n$.
- Let Y be a non-singular level hypersurface $Y = \{P = c\}$, and denote
$$G = Y \cap Q \subset \mathbb{C}^n$$
where c is a regular value of P, and Q is the unit cube in $\mathbb{C}^n$.
- Denote by $\delta > 0$ the distance of Y to the critical points Σ of P.

Problem

We are interested in doubling coverings $\mathcal{U}$ of G in Y, as c approaches a certain critical value of P.

- In this case $\delta \to 0$, and the geometric complexity of Y (in particular, its curvature) near the critical points of P "blows up".

Bounds on doubling coverings

- One can expect that the minimal number $\kappa(\mathcal{U})$ of charts in doubling coverings $\mathcal{U}$ of G in Y also tends to infinity.
- However, this problem turns out to be rather delicate!
- It was shown by Yomdin-Gromov mid80's that for each fixed smoothness k the minimal number of charts in C^k-parameterizations of G is uniformly bounded, in terms of n, d only, independently of c (or δ).

Theorem

There exists a 4-doubling covering $\mathcal{U}$ of G in Y with $\rho(\mathcal{U}) \geq 1/10$ so that

$$\kappa(\mathcal{U}) \leqslant c \log(C/\delta)$$

where $c, C > 0$ depend on n, d and K_P.

Sharpness of the bound

- In some special cases we provide also the lower bound for $\kappa(\mathcal{U})$, of the same order.
- So for doubling coverings, in a strict contrast with C^k-parameterizations, their complexity, at least in some special cases, grows as a logarithm of the distance^{-1} to complex singularities.

Question

This result remains true also for polynomials P with possibly degenerate (and non-isolated) singularities.

Ideas behind the proof

- We provide an explicit construction of a doubling covering $\mathcal{U}$ of G in Y.
- The proof is based on:

  FIRST: The special example of a γ-doubling balls covering of a punctured cube.

  SECOND: A quantitative version of the implicit function theorem (which we also provide) which allows us to produce doubling charts, which are special coordinate charts.

Main steps

- We should think about P as an input to an algorithm.
- It is, as we assume, a generic one, which implies that

$$K_P \cdot \mathrm{dist}(z, \Sigma) \leq \|\nabla P(z)\| \leq nd^4 \mathrm{dist}(z, \Sigma) \qquad z \in Q.$$

  So, now K_P is "given".
- Σ_P is discrete and finite.
- We construct a γ-doubling balls coverings $\mathcal{U}$ of Q_δ with $c \log(C/\delta)$ balls with γ of order $1/K_P$.
- Because of the assumption on P, in each ball of $\mathcal{U}$ we have a lower bound on the norm of $\nabla P(z)$.
- The quantitative implicit function theorem procudes, at a given point z, a coordinate chart of the size proportional to $\|\nabla P(z)\|$.
- Finally, we produce doubling charts by shrinking these coordinate charts according to the requested doubling factor in the theorem.

The result (again)

- Let $P(z)$ be a normalized polynomial on $\mathbb{C}^n$ of degree $d \geq 2$, with isolated and non-degenerate critical points $\Sigma_P = \{w_1, \ldots, w_m\}$, so P satisfies

$$K_P \cdot \mathrm{dist}(z, \Sigma_P) \leq \|\nabla P(z)\| \leq nd^4 \mathrm{dist}(z, \Sigma_P) \qquad z \in Q.$$

  with $K = K_P$.

- Let $Y = \{P(z) = c\}$ be a regular level hypersurface of P. Denote $G = Y \cap Q$, and put $\delta = \mathrm{dist}(G, \Sigma_P) > 0$.

Theorem

There exists a 4-doubling covering $\mathcal{U}$ of G in Y with $\rho(\mathcal{U}) \geq \frac{1}{10}$ and

$$\kappa(\mathcal{U}) \leq \frac{c_{n,d}}{K^{2n}} \log\left(\frac{C_{n,d}}{K\delta}\right)$$

where the constants $c_{n,d}, C_{n,d} > 0$ depend only on n, d.

Doubling inequalities

Doubling-type inequalities

- Let $\Omega \subset G$ be relatively compact domains in Y. Let f be an analytic function in a neighborhood of the closure $\bar{G}$ of G in Y.
- By Doubling-type inequalities we mean inequalities in which we compare the maxima of $|f|$ on a set and its subset

$$\max_G |f| \leq C \max_\Omega |f|.$$

- The doubling constant of f with respect to Ω and G is the ratio

$$DC_f(G,\Omega) = \max_G |f(z)| / \max_\Omega |f(z)|.$$

- Doubling inequalities provide an upper bound on $DC_f(G,\Omega)$ for various classes of analytic functions f on Y.

The doubling constant

- A doubling inequality

$$\max_G |f| \leq C \max_\Omega |f|.$$

- The constant C measures the amount of "information" we have on Ω, and of course it depends on the class of functions to which f belongs.
- There are some natural connections between covering estimates and Doubling-type inequalities.

History & Problem

- In recent years they have been intensively studied for algebraic functions, in connection with various problems in harmonic analysis, potential theory, differential equations, diophantine geometry, probability, complexity, etc.
- However, in these results the variety Y on which the doubling inequalities are considered, is usually fixed, while the degree of the restricted polynomials grows.

Question

An important question of the dependence of the doubling constant on Y remains largely open.

- We show that for polynomials S of degree d_1 restricted to Y we have

$$DC_S(G,\Omega) \leq \left(\frac{c}{\delta}\right)^{C_1},$$

with $c, C_1 > 0$ depending on P, and on the degree d_1 of the restricted polynomial S.

p-valency and doubling inequalities

- We work with algebraic functions. However, it is technically convenient to consider a much larger class of p-valent functions.

Definition

Let $p \in \mathbb{N}$, and let $f(z)$ be an analytic function in a domain $W \subset \mathbb{C}$. The function $f(z)$ is said to be p-valent in W if the equation $f(z) = c$ has at most p roots for any complex c.

- The study of p-valent functions is a classical topic in complex analysis.

Theorem

Let $1 > \alpha > \beta > 0$, and let $f(z)$ be p-valent in the disk D. Then,

$$DC_f(\alpha D, \beta D) \leq ((p+1)\alpha^p + A'_p/(1-\alpha)^{2p+1})/\beta^p =: c_p(\alpha, \beta)$$

where A'_p depends only on P.

Proof

- First, we recall the following classical result of Biernacki '36

Proposition

Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be p-valent in the disk D_1. Then, for any $k \geq p+1$

$$|a_k| \leq A_p \max_{i=0,\dots,p} |a_i| k^{2p-1}$$

where A_p depends only on P.

- By Cauchy formula, applied to βD for any k we have $|a_k| \leq \max_{\beta D} |f| / \beta^k$.
- Hence, by the Proposition for $k \geq p+1$ we get

$$|a_k| \leq A_p k^{2p-1} \max_{\beta D} |f| / \beta^p.$$

Proof (cont.)

- Now, we obtain an upper bound for $|f|$ on αD

$$\max_{\alpha D} |f(z)| \leq \sum_{k=0}^{\infty} |a_k|\alpha^k = \sum_{k=0}^{p} |a_k|\alpha^k + \sum_{k=p+1}^{\infty} |a_k|\alpha^k$$
$$\leq \left(\sum_{k=0}^{p} \frac{\alpha^k}{\beta^k} + \sum_{k=p+1}^{\infty} \frac{A_p k^{2p-1}\alpha^k}{\beta^p} \right) \max_{\beta D} |f|$$
$$= \left(\frac{1-(\alpha/\beta)^{p+1}}{1-\alpha/\beta} + \frac{A_p}{\beta^p} \sum_{k=p+1}^{\infty} k^{2p-1}\alpha^k \right) \max_{\beta D} |f|.$$

- Note that the infinite sum $\sum_{k=p+1}^{\infty} k^{2p-1}\alpha^k$ is in fact a tail of the polylog function $\mathrm{Li}_s(z)$ with the parameters $s = 1-2p, z = \alpha$, which implies

$$\frac{A_p}{\beta^p} \sum_{k=p+1}^{\infty} k^{2p-1}\alpha^k \leq \frac{A'_p}{\beta^p(1-\alpha)^{2p+1}}.$$

where A'_p is another constant, which depends only on P.

Concentric higher-dimensional balls

- An analytic function $f(z)$ on W is called sectionally p-valent, if for each straight line L the restriction f_L of f to $L \cap W$ is p-valent.
- Algebraic functions are sectionally p-valent.

Theorem

Let $1 > \alpha > \beta > 0$. Then, $DC_f(\alpha B, \beta B) \leq c_p(\alpha, \beta)$.

- Let $z \in \alpha B$. Consider the complex straight line L passing through the points 0 and z, and let f_L be the restriction of f to L.
- Now, applying Theorem (previous) to f_L with $\beta B \cap L \subset \alpha B \cap L$, we obtain the required inequality for $f_L(z) = f(z)$

$$|f(z)| \leq \max_{\alpha B \cap L} |f_L| \leq DC_f(\alpha B \cap L, \beta B \cap L) \max_{\beta B \cap L} |f_L| \leq c_p(\alpha, \beta) \max_{\beta B} |f|.$$

Doubling inequalities on non-concentric balls

- The doubling inequality for sectionally p-valent functions can be extended to couples of non-concentric balls.

Corollary

Let $f(z)$ be sectionally p-valent in the ball B_4, and let B_1 be the concentric ball. Let $\Delta_\rho \subset B_1$ be a ball of radius ρ in B_1, not necessarily concentric to it. Then,

$$DC_f(B_1, \Delta_\rho) \leq c_p/\rho^p$$

where $c_p > 0$ depends only on P.

p-valency w.r.t. a certain fixed doubling covering

- We give a very general form of a doubling inequality for analytic functions f on a manifold Y.

Definition

An analytic function f on Y is called sectionally p-valent with respect to the doubling covering $\mathcal{U}$ of Y if for each chart U_j in $\mathcal{U}$ the function $f_j = f \circ \psi_j$ is sectionally p-valent in B_4.

- Certainly, polynomials or algebraic functions on algebraic manifolds Y satisfy this property for each covering $\mathcal{U}$ with algebraic charts, with p depending only on the degrees of the algebraic objects involved.

Doubling inequalities on complex manifolds

- Let Y be a complex manifold, $\Omega \subset G$ be relatively compact domains in Y, and $z \in G$.
- Let f be an analytic function in a neighborhood of $\bar{G}$ in Y, and let $\mathcal{U}$ be a doubling covering of G in Y such that f is sectionally p-valent with respect to $\mathcal{U}$.

Theorem

We have $|f(z)| \leq K(z, \Omega, f) \max_\Omega |f|$ *where*

$$K(z, \Omega, f) = \inf_{Ch \in CH(z, \Omega, \mathcal{U})} \frac{c_p^{\ell(Ch)}}{\tilde{\rho}(U_{j_1}, \Omega)^p \prod_{m=1}^{\ell(Ch)-1} \rho(U_{j_m}, U_{j_{m+1}})^p}$$

and $c_p > 0$ *being the constant from Corollary.*

Main step of the proof

- We do so via continuation along chains of charts in $\mathcal{U}$.
- By the assumptions, for each chart U_j of $\mathcal{U}$ the function $f_j = f \circ \psi_j$ is sectionally p-valent in B_4.
- Let $Ch = \{1, \ldots, n\}$ be a chain joining $z \in G$ and Ω. Clearly, $|f(z)| \leq \max_{U_n} |f|$.
- By the definition of $\rho = \rho(U_j, U_{j+1})$, there is a subball Δ_ρ of radius ρ, such that

$$\Delta_\rho \subset \psi_{j+1}^{-1}(U_j \cap U_{j+1}) \subset B_1.$$

- We have

$$\max_{U_{j+1}} |f| = \max_{B_1} |f_{j+1}| \leq \frac{c_p}{\rho^p} \max_{\Delta_\rho} |f_{j+1}|$$
$$\leq \frac{c_p}{\rho^p} \max_{\psi_{j+1}^{-1}(U_j \cap U_{j+1})} |f_{j+1}| = \frac{c_p}{\rho^p} \max_{U_j \cap U_{j+1}} |f| \leq \frac{c_p}{\rho^p} \max_{U_j} |f|.$$

- This allows us to pass from one chart to the next along the chain.

Some remarks on Harnack-type inequalities

- "Extension along chains" of doubling charts, is one of the classical and widely used tools in study of Harnack-type inequalities for harmonic functions and, more generally, for solutions of certain classes of PDE's.
- There is, however, an essential difference: usually, only coverings with disks are used.
- The reason is that a general complex analytic change of variables preserves harmonic functions only in complex dimension one.
- In the case of two or more variables already linear changes of variables, if not dilations, destroy the condition $\Delta f = 0$.

Some corollaries

- Let f be an analytic function in Y.
- Let $\mathcal{U}$ be a doubling covering, such that f is sectionally p-valent with respect to $\mathcal{U}$.
- Assume that $\rho(\mathcal{U}, \Omega), \rho(\mathcal{U}) \geq \rho$.

Corollary

We have

$$DC_f(G, \Omega) \leq (c_p/\rho^p)^{\kappa(\mathcal{U})}.$$

- Clearly, this inequality can be reversed, obtaining a lower bound on the number of charts in doubling coverings in terms of the doubling constant for certain functions.
- In some special cases, the doubling constant can be directly computed, and thus provide a lower bound for $\kappa(\mathcal{U})$ of order $\log(1/\delta)$ as promised.

Doubling inequality for polynomials on Y

- We obtain an explicit bound in a doubling inequality for polynomials S of degree d_1 on hypersurfaces Y.
- Let $P(z), Y, \Sigma_P, G = Y \cap Q, \delta = \mathrm{dist}(G, \Sigma_P) > 0$ be as before.
- Let $\mathcal{U}$ be the doubling covering of G in Y.
- Let $\Omega \subset G$ be a relatively compact sub-domain of G.
- To simplify the presentation we shall assume that $\rho(\mathcal{U}, \Omega) \geq \frac{1}{10}$.

Corollary

Let f be a restriction of a polynomial S of degree d_1 to Y. Then, we have

$$DC_f(G, \Omega) \leq (c_{n,d}/K_P \delta)^{C_{n,d,d_1}/K_P^{2n}}.$$

Further developments

Doubling chains on complements of level hypersurfaces

- Let P be a polynomial of degree D in $\mathbb{C}^n$, and let $H = \{P = 0\}$.

Theorem

Let $\delta(n, d) \geq \delta > 0$. Then for any $v_1, v_2 \in Q^\delta = Q \setminus H^\delta$ there exists a doubling chain Ch in $Y = \mathbb{C}^n \setminus H$, joining v_1, v_2, with the following properties:

1. *The length of the chain Ch satisfies $l(Ch) \leq 36d \log(180d/\delta) + 1$.*
2. *The intersection radius of the chain Ch satisfies $\rho(Ch) \geq 2^{-d}/3$.*

- The expected bound of order $\log(\frac{1}{\delta})$ for the length of the chain is essential for all the expected applications.
- The constants depend only on n and D, but not on the specific coefficients of the defining polynomial P.

Merci beaucoup.