Expansions of the reals that do not define the natural numbers

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Géométrie et Théorie des Modèles
Paris, 2011

Introduction

Joint work with Philipp Hieronymi and Chris Miller.
Main topic: First order expansions of the real field $\overline{\mathbb{R}}$
(that is, structures of the form $\langle \mathbb{R}, +, \cdot, <, \dots \rangle$).
Sharp divide between:

Wild structures: structures that define the natural numbers;

Restrained structures: structures that do not define $\mathbb{N}$.

Also: certain (i.e., Definably Complete) structures outside $\overline{\mathbb{R}}$.

A topic in between

- Model Theory (especially, o-minimality);
- Descriptive Set Theory;
- Geometric Measure Theory.

Contents

Restrained structures

$\overline{\mathbb{R}} := \langle \mathbb{R}; +, \cdot, < \rangle$ is the field of real numbers.

Definition

An expansion of $\overline{\mathbb{R}}$ is restrained if it does *not* define $\mathbb{N}$, the set of natural numbers.

Examples

- $\overline{\mathbb{R}}$ is restrained (Tarski).
- Any o-minimal structure is restrained (by definition).
- $\langle \overline{\mathbb{R}}, \mathbb{R}^{alg} \rangle$ is restrained, not o-minimal;
  it has o-minimal open core (Robinson, van den Dries).
  ($\mathbb{R}^{alg}$ is the set of real algebraic numbers)
- $\langle \overline{\mathbb{R}}, 2^{\mathbb{Z}} \rangle$ is restrained, not o-minimal; it is d-minimal (vdD).
  ($2^{\mathbb{Z}} = \left\{ 2^n : n \in \mathbb{Z} \right\}$)
- $\langle \overline{\mathbb{R}}, 2^{\mathbb{Z}}, 3^{\mathbb{Z}} \rangle$ does define $\mathbb{N}$ (Hieronymi).

O-minimality

Let $\mathcal{R}$ be an expansion of $\overline{\mathbb{R}}$. $\mathcal{R}$ is o-minimal if every definable subset of $\mathbb{R}$ is a finite union of intervals and points.

$\overline{\mathbb{R}}$ itself is o-minimal (Tarski); $\langle \overline{\mathbb{R}}, \exp \rangle$ is o-minimal (Wilkie).

Every o-minimal structure is restrained.

O-minimal structures are the "tamest" among restrained structures:
every definable set is a finite union of C^1 manifolds,
and every definable function is piecewise C^1.

If C is a definable set, then the

- topological dimension
- Hausdorff dimension
- Minkowski dimension
- any reasonalbe notion of dimension

of C are equal.

O-minimality is a (possible) framework for "tame topology".

The wilderness

If $\mathcal{R}$ defines $\mathbb{N}$, then it defines every projective set;
in particular, it defines all closed sets and all continuous functions:
all "tameness" is lost!

Beyond o-minimality

People (C. Miller, in particular) looked for notions of “topological tameness” (for structures on $\mathbb{R}$) that would be weaker than o-minimality, but still useful. Several possible notions emerged. Among the most popular, we have the following ones.

- $\mathcal{R}$ is d-minimal if every definable subset C of $\mathbb{R}$ is the union of an open set and finitely many discrete sets, where the number of discrete sets does not depend on the parameters of definition of C.
- $\mathcal{R}$ has o-minimal open core if the open core of $\mathcal{R}$ o-minimal.

The open core of $\mathcal{R}$ is the structure on $\mathbb{R}$ given by all $\mathcal{R}$-definable open sets (if $\mathcal{R}$ is o-minimal, then it coincides with its own open core).

We will focus on two notions (being restrained and i-minimality), which are

- natural
- general
- powerful.

Basic theorem

The starting point for the study of restrained structures is:

Theorem (Hieronymi 2010)

An expansion of $\overline{\mathbb{R}}$ is restrained iff
For every definable discrete set $D \subset \mathbb{R}^n$ and every definable function $f : D \to \mathbb{R}$, $f(D)$ is nowhere dense in $\mathbb{R}$.

Almost all proofs about restrained structures
do not use the definition directly, but instead rely on
the equivalent characterization given by Hieronymis's Theorem.

Basic theorem

The starting point for the study of restrained structures is:

Theorem (Hieronymi 2010)

An expansion of $\overline{\mathbb{R}}$ is restrained iff
For every definable discrete set $D \subset \mathbb{R}^n$ and every definable function $f : D \to \mathbb{R}$, $f(D)$ is nowhere dense in $\mathbb{R}$.

Example

The structure $\langle \overline{\mathbb{R}}, 2^{\mathbb{Z}}, 3^{\mathbb{Z}} \rangle$ defines $\mathbb{N}$.

Proof.

The set $2^{\mathbb{Z}} \times 3^{\mathbb{Z}} \subset \mathbb{R}^2$ is discrete.
Its image $\left\{ 2^n \cdot 3^m : n, m \in \mathbb{Z} \right\}$ is dense in $\mathbb{R}_{\geq 0}$. □

Main properties

Closed sets

$\mathcal{R}$ is a restrained expansion of $\overline{\mathbb{R}}$.

Theorem

Let $C \subseteq \mathbb{R}^n$ be closed and definable. Then,

- *the topological dimension*
- *and the Hausdorff dimension*

of C are equal.

Definition (Mandelbrot)

A fractal set is a set whose Hausdorff dimension and topological dimension do not coincide.

Closed fractal sets are not definable in restrained structures.

Main properties

Continuous functions

$\mathcal{R}$ is a restrained expansion of $\overline{\mathbb{R}}$.

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be definable and continuous. Then,

- *f is C^1 outside a closed set of dimension less than n.*
- *If f is C^1, then the set of singular values of f has dimension less than m.*

Restrained vs. wild

- Let $\mathcal{R}$ be restrained.
  The Peano curve is not definable; more generally, a continuous surjection between $\mathbb{R}^n$ and $\mathbb{R}^{n+1}$ is not definable.
- Assume that $\mathcal{R}$ does define the natural numbers.
  Then, every closed (and every projective) set and every continuous function are definable.

Interpreting $\mathbb{N}$ in a restrained structure

If $\mathcal{R}$ is o-minimal, then every definable set is a finite union of embedded manifolds.

A restrained structure can still define some "pathological" sets.

Let $C \subseteq \mathbb{R}$. $\langle \overline{\mathbb{R}}, C^{\#} \rangle$ is the expansion of $\overline{\mathbb{R}}$ by all subsets of C^n.

Theorem (Friedman et al.)

There exists a compact perfect nowhere dense nonempty set $C \subseteq \mathbb{R}$, such that $\langle \overline{\mathbb{R}}, C^{\#} \rangle$ is restrained (and even i-minimal).

Corollary

There exists a restrained (and even i-minimal) structure that defines a structure Borel isomorphic to $\langle \mathbb{R}; \mathbb{N}, +, \cdot, < \rangle$.

Proof.

Take any C as in the previous theorem.

C contains a subset that is Borel isomorphic to $\mathbb{R}$. □

“Bad” sets definable in restrained structures

Theorem (van den Dries)

Let F be a real closed subfield of $\mathbb{R}$.
Then, $\langle \overline{\mathbb{R}}, F \rangle$ has o-minimal open core (and, in particular, it is restrained).

F is a Bernstein subset of $\mathbb{R}$ if
F and $\mathbb{R} \setminus F$ intersect every closed uncountable subset of $\mathbb{R}$.
A Bernstein set is not Lebesgue measurable, and does not have the property of Baire.
A set is projective if it is definable in $\langle \overline{\mathbb{R}}, \mathbb{N} \rangle$.

Exercise

There exists F a real closed subfield of $\mathbb{R}$ that is a Bernstein set and not projective.

Corollary

There exist structures with o-minimal open core that define sets that are Bernstein, or that are not projective.

I-minimal structures

A stronger tameness notion.

Definition-Theorem

Let $\mathcal{R}$ be an expansion of $\overline{\mathbb{R}}$. T.f.a.e.:

1. Every definable subset of $\mathbb{R}$ has interior or it is nowhere dense.
2. Every definable subset of $\mathbb{R}$ has interior or it is Lebesgue null.
3. Every definable subset of $\mathbb{R}$ has interior or it has Hausdorff dimension 0.
4. Every definable subset of $\mathbb{R}$ has interior or it has Minkowski dimension 0.

If it satisfies any of the above conditions, $\mathcal{R}$ is i-minimal.

Examples

Examples

- $\overline{\mathbb{R}}$, and any o-minimal structure, is i-minimal;
- $\langle \overline{\mathbb{R}}, 2^{\mathbb{Z}} \rangle$, and any d-minimal structure, is i-minimal;
- $\langle \overline{\mathbb{R}}, \mathbb{R}^{alg} \rangle$ is restrained but not i-minimal;
- Any i-minimal structure is restrained.

Theorem

Let $C \subseteq \mathbb{R}$ be a closed set with empty interior.
If $\langle \overline{\mathbb{R}}, C \rangle$ is restrained, then $\langle \overline{\mathbb{R}}, C^{\#} \rangle$ is i-minimal.

Dimension

$\mathbb{K}$ is a structure with a topology.

Definition

Given $X \subseteq \mathbb{K}^n$, the dimension of X is $\dim(X)$, the maximum $d \in \mathbb{N}$ such that there is a projection onto a d-dimensional coordinate space $\mathbb{K}^d$, such that $\pi(X)$ has nonempty interior.

Example

- If $\mathbb{K}$ is o-minimal, then $\dim(X)$ is the usual o-minimal dimension.
- If $\mathbb{K}$ is a pure algebraically closed field (with the Zariski topology), then $\dim(X)$ is the algebraic dimension.
- If $\mathbb{K}$ is a p-adic field, or an algebraically closed field, then $\dim(X)$ is the "usual" dimension for such fields.

Properties of i-minimal structures

Recall:

Corollary

There exists a structure with o-minimal open core that defines a Bernstein set.

Properties of i-minimal structures

Theorem

Let $\mathcal{R}$ be i-minimal.

1. *Let $C \subseteq \mathbb{R}^n$ be definable. Then:*
   - *C is Lebesgue measurable;*
   - *C has the property of Baire;*
   - $\dim(C) = \dim(\overline{C})$*;*
   - $\dim(C) = \dim_H(C)$*;*
2. *Every definable function $f : \mathbb{R}^n \to \mathbb{R}^m$ is $\mathcal{C}^1$ outside a closed nowhere dense set.*

$\dim_H(C)$ is the Hausdorff dimension of C.
We don't know if we can also say that $\dim(C)$ is equal to the topological dimension of C.

Outside $\mathbb{R}$

Definition

A structure $\mathbb{K}$ expanding an ordered field is Definably Complete (DC) if every definable subset of $\mathbb{K}$ has a least upper bound in $\mathbb{K} \cup \{\pm\infty\}$.

$\mathbb{K}$ will always be a DC structure (expanding an ordered field).

Remark

1. $\mathbb{K}$ is a real closed field;
2. If $C \subseteq \mathbb{K}^n$ is d-compact
   (that is, closed, definable, and bounded)
   and $f : C \to \mathbb{K}^m$ is definable and continuous,
   then $f(C)$ is also d-compact.

Definably Baire structures

Recall:

Theorem (Baire Category Theorem)

$\mathbb{R}$ is not meager in itself:
That is, $\mathbb{R}$ is not the union of an increasing family $\left(C_t : t \in \mathbb{R}\right)$ of nowhere dense subsets of $\mathbb{R}$.

Definably Baire structures

Theorem (P. Hieronymi)

$\mathbb{K}$ *is* *Definably Baire*:
That is, $\mathbb{K}$ *is not the union of a definable increasing family* $\left(C_t : t \in \mathbb{K}\right)$ *of nowhere dense subsets of* $\mathbb{K}$.

Corollary (F–Servi)

Let $\mathbb{K} = \langle K; +, \cdot, <, \exp\rangle$ *be an ordered exponential field.*
If $\mathbb{K}$ *is DC, then it is o-minimal.*

An ordered exponential field is an ordered field with an increasing homomorphism $\exp : \langle \mathbb{K}, +\rangle \to \langle \mathbb{K}_{>0}, \cdot\rangle$.

Restrained DC structures

Recall:

Theorem (P. Hieronymi)

An expansion of $\overline{\mathbb{R}}$ is restrained iff:
For every definable discrete set $D \subset \mathbb{R}^n$ and every definable function $f : D \to \mathbb{R}$, $f(D)$ is nowhere dense in $\mathbb{R}$.

Restrained DC structures

Definition

$\mathbb{K}$ is restrained if it is DC and:
For every definable discrete set $D \subset \mathbb{K}^n$ and every definable function $f : D \to \mathbb{K}$, $f(D)$ is nowhere dense in $\mathbb{K}$.

Example

1. Every o-minimal structure is restrained.
2. Every expansion of $\overline{\mathbb{R}}$ is DC (but not necessarily restrained).

Work in progress...

Tautology

Either $\mathcal{R}$ is restrained, or it defines $\mathbb{Z}$.

Work in progress...

Theorem (almost. . .)

Either $\mathbb{K}$ is restrained, or it defines a discrete subring.

Work in progress...

Theorem (almost. . .)

Either $\mathbb{K}$ is restrained, or it defines a discrete subring.

Remark

If a discrete subring Z is definable in $\mathbb{K}$, then

1. Z is unique;
2. $Z_{\geq 0}$ is a model of second order Peano arithmetic:
   elements of $\mathbb{K}$ code definable subsets of $Z_{\geq 0}$, in the same way as real numbers code (via, e.g., their binary representation) sets of natural numbers.
   Therefore, many results from the reals can be transferred to $\mathbb{K}$.

Properties

Theorem

Let $\mathbb{K}$ be restrained. Let $f : \mathbb{K}^n \to \mathbb{K}^m$ be a definable map.

- *If f is continuous, then, outside a nowhere dense set, f is C^1.*
- *If f is C^1, then the set of singular values of f is nowhere dense.*
- *If $f : \mathbb{K} \to \mathbb{K}$ is monotone, then, outside a nowhere dense set, f is C^1.*

Properties

Theorem

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Theorem (Lebesgue)

Let $f : \mathbb{R} \to \mathbb{R}$ be a monotone function.
Then, f is differentiable almost everywhere.

Properties

Theorem

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- *If f is continuous, then, outside a nowhere dense set, f is C^1.*
- *If f is C^1, then the set of singular values of f is nowhere dense.*
- *If $f : \mathbb{K} \to \mathbb{K}$ is monotone, then, outside a nowhere dense set, f is C^1.*

Corollary (almost...)

Let $\mathbb{K}$ be a DC structure. Let $f : \mathbb{K} \to \mathbb{K}$ be definable and monotone. Then, the derivative f' exists on a dense subset of $\mathbb{K}$.

Proof.

If $\mathbb{K}$ is restrained, see above.
If $\mathbb{K}$ is not restrained, then it is a model of second order Peano arithmetic, and we can use measure theory to prove the result. □

Good dimension

Definition

A good dimension is a function d from definable sets to $\mathbb{N} \cup \{-\infty\}$, such that:

- $d(\mathbb{K}) = 1, \quad d(\{a\}) = 0$ for every $a \in \mathbb{K}$;
- $d(A) = -\infty$ iff $A = \emptyset$;
- $d(A^\sigma) = d(A)$, for every permutation of coordinates σ;
- Union: $d(A \cup B) = \max\big(d(A), d(B)\big)$.

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- Union: $d(A \cup B) = \max\bigl(d(A), d(B)\bigr)$.

Given $A \subseteq \mathbb{K}^{n+1}$ definable and $\pi : \mathbb{K}^{n+1} \to \mathbb{K}^n$ projection:

- The set $\Bigl\{x \in \mathbb{K}^n : d(A_x) = 0\Bigr\}$ is definable;
- Fubini: If $d(A_x) \leq 0$ for every $x \in \mathbb{K}^n$, then $d(A) = d(\pi(A))$;
  If $d(A_x) = 1$ for every $x \in \pi(A)$, then $d(A) = d(\pi(A)) + 1$.

Example

If $\mathbb{K}$ is o-minimal (or d-minimal), then dim is a good dimension.

Conjecture

If $\mathbb{K}$ is restrained, then there is a (unique) good dimension, which coincides with dim on closed definable sets.

Example

If $\mathcal{R} = \langle \overline{\mathbb{R}}, \mathbb{R}^{alg} \rangle$, then dim is not a good dimension on $\mathcal{R}$ (Union Axiom fails), but there is a good dimension, which coincides with dim on closed set.

D_Σ sets

$\mathbb{K}$ is a DC structure.

Definition

Let $C \subseteq \mathbb{K}^{n+m}$ be a closed definable set.
Its projection $\pi(C) \subseteq \mathbb{K}^n$ is a D_Σ set.

Remark

The family of D_Σ sets is closed under

- finite intersections,
- finite unions,
- Cartesian products,
- images under continuous definable function,
- but not complement.

It includes all open and all closed definable sets.

Dimension on D_Σ sets

$\mathbb{K}$ is restrained

Idea: dim is a good dimension, but only on D_Σ sets.

Theorem

A and B are D_Σ sets.

- $\dim(\overline{A}) = \dim(A)$;
- $f : A \to \mathbb{K}^n$ *definable and continuous* $\Rightarrow \dim(f(A)) \leq \dim(A)$;
- *Union:* $\dim(A \cup B) = \max\big(\dim(A), \dim(B)\big)$;

Given $A \subseteq \mathbb{K}^{n+m}$ D_Σ, $\pi : \mathbb{K}^{n+m} \to \mathbb{K}^n$ *projection, and* $p \in \mathbb{N}$:

- *The set* $\left\{ x \in \mathbb{K}^n : \dim(A_x) = p \right\}$ *is definable;*
- *Fubini: If* $\dim(A_x) = p$ *for every* $x \in \pi(A) \Rightarrow \dim(A) = \dim(\pi(A)) + p$.

Most of the results for "closed definable" sets remain true for D_Σ.

Proposition

Let $\mathcal{R}$ be a restrained expansion of $\overline{\mathbb{R}}$, and $D \subseteq \mathbb{R}^n$ be a D_Σ sets. T.f.a.e.:

1. *D has empty interior;*
2. *D is nowhere dense;*
3. *D is null (i.e., it has Lebesgue measure 0).*

Moreover, $\dim_H(D) = \dim(D) = \dim(\overline{D})$, and $\dim(D)$ is equal to the topological dimension of D.

Proposition

Let $\mathbb{K}$ be restrained and $D \subseteq \mathbb{K}^n$ be a D_Σ sets. T.f.a.e.:

1. *D has empty interior;*
2. *D is nowhere dense.*

Moreover, $\dim(D) = \dim(\overline{D})$.

Matroid associated to dim

$\mathbb{K}$ is ω-saturated, and $A \subset \mathbb{K}$ is any set.

Definition

$b \in \mathrm{Fcl}(A)$ iff there exists $a \in A^n$ and a $\emptyset$-definable D_Σ set $C \subseteq \mathbb{K}^{n+1}$, such that

- $\langle a, b \rangle \in C$;
- for every $x \in \mathbb{K}^n$, $\dim(C_x) \leq 0$.

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Proposition

Fcl *is a finitary matroid.*

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- for every $x \in \mathbb{K}^n$, $\dim(C_x) \leq 0$.

Proposition

Fcl *is a finitary matroid.*

Let rk *be the rank associated to* Fcl. *Given a $\emptyset$-definable D_Σ set D,*

$$\dim(D) = \max\Big\{ \mathrm{rk}(b) : b \in D \Big\}.$$

Matroid associated to dim

$\mathbb{K}$ is ω-saturated, and $A \subset \mathbb{K}$ is any set.

Definition

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Proposition

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$$\dim(D) = \max\Big\{ \mathrm{rk}(b) : b \in D \Big\}.$$

Example

If $\mathbb{K}$ is o-minimal, then Fcl is the algebraic closure.

D_Σ-Uniformization

Lemma

Let $C \subseteq \mathbb{K}^{n+m}$ be a D_Σ set, and $\pi : \mathbb{K}^{n+m} \to \mathbb{K}^n$ be the projection. There exists a definable function $f : \pi(C) \to \mathbb{K}^m$, such that $\Gamma(f) \subseteq C$. Moreover, f can be taken to be continuous off a nowhere dense subset of $\mathbb{K}^n$.

Proof.

1. Easy for closed sets.
2. Reduce from D_Σ to closed sets. □

Corollary

Let $g : \mathbb{K}^n \to \mathbb{K}^m$ be a definable continuous function. Then, there exists a definable function $f : \mathrm{Im}(g) \to \mathbb{K}^m$, such that $g \circ f = \mathbf{1}_{\mathrm{Im}(g)}$, and f is continuous off a nowhere dense set.

Unary sets

$\mathcal{R}$ is a restrained expansion of $\overline{\mathbb{R}}$.

Theorem

Let $C \subseteq \mathbb{R}^n$ be closed and definable. Then,

- *the dimension*
- *the topological dimension*
- *the Hausdorff dimension*

of C are equal.

Unary sets

$\mathcal{R}$ is a restrained expansion of $\overline{\mathbb{R}}$.

Theorem

Let $C \subseteq \mathbb{R}$ be closed and definable. Then,

- *the dimension*
- *the topological dimension*
- *the Hausdorff dimension*
- *and the upper Minkowski dimension*

of C are equal.

Sketch of the proof

Theorem

Let $C \subseteq \mathbb{R}$ be closed and definable. Let d be a "reasonable" fractal dimension. Then, $d(C) = \dim(C)$.

Proof.

- If $\dim(C) = 1$, then C has nonempty interior, and thus $d(C) = 1$.

Sketch of the proof

Theorem

Let $C \subseteq \mathbb{R}$ be closed and definable. Let d be a "reasonable" fractal dimension. Then, $d(C) = \dim(C)$.

Proof.

- If $\dim(C) = 1$, then C has nonempty interior, and thus $d(C) = 1$.
- Assume, for a contradiction, that $\dim(C) = 0$, but $d(C) > 0$.
  Let $n \in \mathbb{N}$ large enough, such that $d(C^n) \gg 1$.
  Let $T : \mathbb{R}^n \to \mathbb{R}$ be a linear function, such that $d(E^2) > 1$,
  where $E := T(C^n)$.

Proof

...Continued

Memo: $E \subset \mathbb{R}$, $d(E^2) > 1$, but $\dim(E) = 0$.
Let $Q(E) := \left\{ \frac{a-b}{c-d} : a, b, c \neq d \in E \right\}$.
Claim: $Q(E)$ is dense in $\mathbb{R}$.
It is easy to see that the above claim contradicts the fact that $\mathcal{R}$ is restrained.

Remark (Falconer)

$Q(E)$ is the set of slopes of nonvertical lines connecting pair of points in E^2.

Thus, if $Q(E)$ were not dense in $\mathbb{R}$, the set of difference $\left\{ u - v : u, v \in E^2 \right\}$ would be disjoint from some double cone $V \subset \mathbb{R}^3$.
After a rotation in $\mathbb{R}^3$, we can assume that V is vertical:
thus, E^2 is the graph of a partial Lipschitz function $f : \mathbb{R} \to \mathbb{R}$;
in particular, $d(E^2) \leq 1$, contradiction. □

A conjecture in GMT

Denote:

$\dim_M$: upper Minkowski dimension.

$\dim_H$: Hausdorff dimension.

Conjecture

Assume: $d < n$ are natural numbers, X is a compact subset of $\mathbb{R}^n$, such that $\dim_M(X) > d$.
Then: There exist natural numbers k and m, and a rational function $g : \mathbb{R}^{nk} \to \mathbb{R}^m$, such that

$$\dim_H\left(\overline{g(X^k)}\right) > kd.$$

Moreover: k depends only on d and n (and not on X).

The above conjecture would imply that $\dim_H(C) = \dim_M(C) = \dim(C)$ for any closed set definable in a restrained structure.

Conjectures

$\mathcal{R}$ restrained expansion of $\overline{\mathbb{R}}$.

1. Let C be closed and definable.
   Then, $\dim_M(C) = \dim_H(C) = \dim(C)$.
   (True for unary sets).
2. There is no surjective definable function from $\mathbb{R}$ to $\mathbb{R}^2$.
   (True for continuous functions).
3. There is a good dimension d,
   which coincides with dim on D_Σ sets.
   (It implies Conjecture **??**).
4. If $\mathcal{R}$ is i-minimal, then dim is a good dimension.
   (True when $\mathcal{R}$ has Definable Skolem Functions).
5. The open core of $\mathcal{R}$ is i-minimal.