Cohomology of algebraic varieties over non-archimedean fields

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- Joint with Mário Edmundo (Lisbon U.) and Jinhe Ye (Sorbonne U.);
- Builds on:
  - E. Hrushovski and F. Loeser's tame topology over non archimedean fields;
  - ideas from previews work on cohomology in o-minimal structures (M. Edmundo, N. Peatfield, G. Jones, L. Prelli, ...).

Some notation

For the rest of the talk (and unless otherwise stated)

- we let K be an algebraically closed field;
- we let Var_K be the category of separated schemes of finite type over K;
- most results hold in general for $V \in \mathrm{Var}_K$ (and, in some cases, we will need V to be quasi-projective);
- However, in spite of simplicity during the exposition, V will be an affine variety over K, $V = \mathrm{Spec}(K[T]/(f))$, where $T = (T_1, \ldots, T_n)$ and $f = (f_1, \ldots, f_m)$ are polynomials in $K[T]$.

Classical setting $K = \mathbb{C}$

One of the initial goals of algebraic geometry was to have a better understanding of the set of $\mathbb{C}$-points of V, in our case:

$$V(\mathbb{C}) = \{x \in \mathbb{C}^n : f_1(x) = \ldots = f_m(x) = 0\}.$$

A crucial point in the study of $V(\mathbb{C})$ is to endow it with the Euclidean topology, namely, the topology induced by the archimedean norm

$$|\ \ | : \mathbb{C} \to \mathbb{R}_{\geq 0}$$

We get the best of two worlds: $\mathbb{C}$ is algebraically closed, but also complete with respect to this norm.

Classical setting $K = \mathbb{C}$

So one sets a functor

$$\begin{aligned} \mathrm{Var}_{\mathbb{C}} &\to \mathrm{Top} \\ V &\to V^{\mathrm{an}} \end{aligned} \qquad \text{(actually to } \mathrm{AnSpaces}_{\mathbb{C}}\text{)}$$

where $V^{\mathrm{an}} = V(\mathbb{C})$ equipped with the Euclidean topology.

Away from a small set of singularities, V^{an} is a complex analytic manifold, so in particular, a very nice locally compact topological space.

Classical setting $K = \mathbb{C}$

One can thus use (sheaf) cohomology groups to further study varieties over $\mathbb{C}$. In fact most cohomology theories coincide in this case:

$$H^*_{\mathrm{sing}}(V^{\mathrm{an}}; \mathbb{C}) \simeq H^*(V^{\mathrm{an}}; \mathbb{C}) \simeq H^*_{dR}(V^{\mathrm{an}})$$

Classical setting $K = \mathbb{C}$

Two theorems about the analytification functor:

Theorem (Hironaka/...)

(A) V^{an} *admits a deformation retraction to a finite simplicial complex.*

(B)

- $H^p(V^{\mathrm{an}}; L)$ *are finitely generated;*
- $H^p(V^{\mathrm{an}}; L) = 0$ *for* $p > 2\dim V$.

(here L is a finitely generated module over a noetherian ring)

Note: The same holds for cohomology with compact supports

$$H_c^*(V^{\mathrm{an}}; L)$$

...

But, actually what was wrong with V all along in an arbitrary algebraically closed field K? We still have a functor, namely the forgetful functor

$$\mathrm{Var}_K \to \mathrm{Top}$$

which sends V to its underlying topological space (i.e. with the Zariski topology).

The problem is that this way we do not obtain nice topological spaces... but

Grothendieck sites

Instead of defining a functor from Var_K to Top, we associate to each element in Var_K a site

$$\mathrm{Var}_K \rightsquigarrow \mathrm{Sites}$$

together with induced sheaf cohomology groups associated to such sites.

The map that makes everything work here associates to $V \in \mathrm{Var}_K$ its étale site V_{et}.

Grothendieck sites

Using the étale sites one defines the étale cohomology and shows

Theorem (Grothendieck, Artin)

(B)

- $H_c^p(V_{et}; L)$ *are finite;*
- $H_c^p(V_{et}; L) = 0$ *for* $p > 2\dim V$*;*
- *If* K' *is an algebraically closed extension of* K*, then*

$$H_c^*(V_{et}; L) \simeq H_c^*(V_{K',et}; L)$$

- *If* $K = \mathbb{C}$*, then*

$$H_c^*(V_{et}; L) \simeq H_c^*(V^{\mathrm{an}}; L) \text{ and } H^*(V_{et}; L) \simeq H^*(V^{\mathrm{an}}; L)$$

(here L *is a finite field)*

Note: finiteness and invariance hold without supports.

What about non-archimedean fields?

Suppose now K comes equipped with a (non-archimedean) norm $|\ \ | : K \to \mathbb{R}_{\geq 0}$, i.e.:

- $|x| = 0 \quad \text{iff} \quad x = 0$;
- $|x \cdot y| = |x| \cdot |y|$;
- $|x + y| \leq \max\{|x|, |y|\}$ (ultrametric inequality)

and assume it is complete with respect to this norm.

Examples:

- Any algebraically closed field K with the trivial norm
- $\mathbb{C}_p$ (the completion of the algebraic closure of $\mathbb{Q}_p$)
- Hahn series fields like $\overline{\mathbb{F}_p}^{\mathrm{alg}}((t^{\mathbb{Q}}))$.

...

Although it is tempting to carry over all the definitions from $\mathbb{C}$ to K, and in particular define V^{an} as $V(K)$ together with the topology induced by the norm, here the topological spaces we get are again terrible... (balls are clopen, and we get totally disconnected spaces, not necessarily locally compact, etc.).
Alternatives:

- Tate
- Raynaud
- Berkovich

Berkovich analytification

- Define V^{an} as

$$V^{\mathrm{an}} = \{(x, |\ |_x) : x \in V \text{ and } |\ |_x : k(x) \to \mathbb{R}_{\geq 0}\}$$

  where $|\ |_x$ extends $|\ | : K \to \mathbb{R}_{\geq 0}$.
- Put on V^{an} the coarsest topology making all $x \mapsto |f|_x$ continuous for all $f \in \mathcal{O}_V(U)$ and all $U \subseteq V$ Zariski open.

V^{an} is a nice locally compact topological space!

Berkovich analytification

V. Berkovich shows:

Theorem (Berkovich)

(A) *(Under extra assumptions)* V^{an} *admits a deformation retraction to a finite simplicial complex.*

(B)

- $H_c^p(V^{\mathrm{an}}; L)$ *are finitely generated;*
- $H^p(V^{\mathrm{an}}; L) = 0$ *for* $p > \dim V$*;*
- *There is a finite Galois extension* $K \leq K'$ *such that for any non-archimeadean field extension* $K' \leq K''$ *we have*

$$H_c^*(V_{K'}^{\mathrm{an}}; L) \simeq H_c^*(V_{K''}^{\mathrm{an}}; L)$$

(here L is a finitely generated module over a noetherian ring)

Note: finiteness and invariance holds without supports.

What about more general valued fields?

Assume K is now endowed with a valuation of arbitrary rank, i.e.,

$$\mathrm{val}\colon K \to \Gamma_{K,\infty}$$

- $\mathrm{val}(x) = \infty \quad \text{iff} \quad x = 0$;
- $\mathrm{val}(x \cdot y) = \mathrm{val}(f) + \mathrm{val}(g)$;
- $\mathrm{val}(x + y) \geq \min\{\mathrm{val}(x), \mathrm{val}(y)\}$

where $\Gamma_K = (\Gamma_K, 0, +, <)$ is an ordered abelian (divisible) group and ∞ is such that $\gamma < \infty$ for all $\gamma \in \Gamma_K$.

...

As before (in the rank 1 case), the valuation topology does not provide nice topological spaces (balls remain clopen, spaces are totally disconnected, not locally compact)...
but...

Hrushovski and Loeser setting

E. Hrushovski and F. Loeser introduced a model-theoretic replacement of V^{an} over (K, val), and they defined a functor

$$\mathrm{Var}_K \to \mathrm{Top}$$
$$V \to \widehat{V}(K)$$

which shares many properties with Berkovich's analytification functor.
However, $\widehat{V}$ it is still kind of a <u>bad</u> topological space (e.g. <u>not</u> locally compact)...

(Part of the magic): (pro)-definable analogues of topological notions behave well.

Hrushovski and Loeser setting

Their main result shows that:

Theorem (Hrushovski-Loeser)

(A) *(V quasi-projective) $\widehat{V}(K)$ admits a pro-definable (continuous) deformation retraction to a definable subset of $\Gamma_{K,\infty}$.*

(A') *If K is a complete rank one valued field and $K^{\max}$ is maximally complete extension of K with value group $\mathbb{R}$, then:*

- *$V^{\mathrm{an}}_{K^{\max}}$ is canonically homeomorphic to $\widehat{V}(K^{\max})$;*

Our work

Unlike in V^{an} there is no useful topological cohomology in $\widehat{V}(K)$ ... What about theorem (B)?

We introduced a site in $\widehat{V}(K)$, the $\widehat{\mathrm{v+g}}$-site and show for the associated $\widehat{\mathrm{v+g}}$ cohomology:

Theorem (CK-Edmundo-Ye)

(B)

- $H_c^p(\widehat{V}; L)$ *are finitely generated;*
- $H_c^p(\widehat{V}; L) = 0$ *for* $p > \dim V$*;*
- *If* K' *is an algebraically closed valued field extension of* K*, then*

$$H_c^*(\widehat{V}(K); L) \simeq H_c^*(\widehat{V}(K'); L)$$

(here L *is a finitely generated module over a noetherian ring)*

Our work

In addition, if K is a complete rank one valued field, we have

$$H_c^*(\widehat{V}(K^{\max}); L) \simeq H_c^*(\widehat{V}(K^{\max})_{\mathrm{top}}; L) \simeq H_c^*(V^{\mathrm{an}}_{K^{\max}}; L)$$

$\widehat{V}$ in 10 minutes... (with a bit of hope)

- The language of valued fields $\mathcal{L}_{\mathrm{div}}$ is the language of rings $(+,-,\cdot,0,1)$ together with a binary relation div interpreted in K by
$$\mathrm{div}(x,y) \Leftrightarrow \mathrm{val}(x) \leq \mathrm{val}(y).$$
- We work more generally in $\mathcal{L} = \mathcal{L}_{\mathrm{div}}^{\mathrm{eq}}$, so by definable we mean $\mathcal{L}_{\mathrm{div}}^{\mathrm{eq}}$-definable (with parameters).
- In particular, the value group $\Gamma_K \cong K^\times/(K^\circ)^\times$ and the residue field $k_K = K^\circ/K^{\circ\circ}$ are definable.
- We fix some sufficiently large elementary extension $(\mathcal{U}, \mathrm{val})$ of (K, val) in which everything will happen (monster model).
- For x a tuple of variables, $S_x(\mathcal{U})$ is the space of types over $\mathcal{U}$. Such types are called *global types*.

$\widehat{V}$ in 10 minutes... (with a bit of hope)

- A type $p \in S_x(K)$ is orthogonal to Γ if there is a realization $a \in L$ of p in an elementary extension L of K such that $\Gamma_L = \Gamma_K$.
- Let $A \subseteq K$. A type $p \in S_x(K)$ is A-definable if for every $\mathcal{L}$-formula $\varphi(x, y)$, there is an $\mathcal{L}(A)$-formula $\psi(y)$ such that for every $b \in K^{|y|}$

$$\varphi(x, b) \in p \Leftrightarrow K \models \psi(b).$$

Definition

Let D be a definable subset of K^n (for example $V(K)$). The *stable completion of D over K*, denoted $\widehat{D}(K)$, is the set of global K-definable types concentrating on D which are orthogonal to Γ_K.

$\widehat{V}$ in 10 minutes... (with a bit of hope)

Definition

Let D be a definable subset of K^n. The *stable completion of X over K*, denoted $\widehat{D}(K)$, is the set of global K-definable types concentrating on D which are orthogonal to Γ_K.

- Given a K-definable function $f \colon D \to \Gamma_K$, there is an induced map $f_* \colon \widehat{D}(K) \to \Gamma_K$ sending p to $f_*(a)$ for some (any) realization a of p.
- Given an algebraic variety V over K, we let $\widehat{V}(K)$ be $\widehat{D}(K)$ where $D = V(K)$.
- We endow $\widehat{V}(K)$ with the following topology given by the subbasis of sets of the form

$$\{p \in \widehat{U}(K) : v(f)_*(p) \in I\}$$

where U is a basic open subset of V, $f \in \mathcal{O}_V(U)$, $v(f)$ is the (coordinate) composition with the valuation and I is an open definable subset of $\Gamma_{K,\infty}$.

$\widehat{V}$ in 10 minutes... (with a bit of hope)

Theorem (Hrushovski-Loeser)

- *$\widehat{V}(K)$ is (functorially) a strict pro-definable set.*
- *V is separated if and only if $\widehat{V}(K)$ is Hausdorff.*
- *V is complete if and only if $\widehat{V}(K)$ is definably compact.*
- *(V quasi-projective) $\widehat{V}(K)$ admits a pro-definable (continuous) deformation retraction to a definable subset of $\Gamma_{K,\infty}$ (piece-wise linear).*
- *Suppose $\Gamma_K \subseteq \mathbb{R}$ (rank 1). Then there is a deformation retraction of $\widehat{V}(K)$ to a finite simplicial complex (no smoothness assumption on V).*

The $\mathrm{v+g}$ and $\widehat{\mathrm{v+g}}$-sites

A subset of $V(K)$ is called a *basic* $\mathrm{v+g}$-*open subset* if it is of the form

$$\bigcap_{j\in J}\{u \in U_j(K) : \mathrm{val}(h_j(u)) < \mathrm{val}(g_j(u))\}$$

where J is a finite set and $h_j, g_j \in \mathcal{O}_V(U_j)$ are regular functions on a Zariski open subset $U_j \subseteq V$. A subset of $V(K)$ is a $\mathrm{v+g}$-*open subset* if it is a finite union of basic $\mathrm{v+g}$-open subsets; it is $\mathrm{v+g}$-*closed* if it is the complement of a $\mathrm{v+g}$-open subset.

The $\mathrm{v+g}$-*site on* $V(K)$, denoted $V_{\mathrm{v+g}}$, is the category $\mathrm{Op}(V_{\mathrm{v+g}})$ whose objects are the $\mathrm{v+g}$-open subsets of $V(K)$, the morphisms are the inclusions and the admissible covers $\mathrm{cov}(U)$ of $U \in \mathrm{Op}(V_{\mathrm{v+g}})$ are covers by $\mathrm{v+g}$-open subsets of V with finite subcovers.

The $\mathrm{v+g}$ and $\widehat{\mathrm{v+g}}$-sites

Fact

Let W be a definable subset of $V(K)$. Then W is $\mathrm{v+g}$-open (resp. $\mathrm{v+g}$-closed) if and only if $\widehat{W}(K)$ is open (resp. closed) in $\widehat{V}(K)$. Moreover, a basis for the topology on $\widehat{V}(K)$ is given by

$$\{\widehat{U} : U \in \mathrm{Op}(V_{\mathrm{v+g}})\}.$$

Definition

The $\widehat{\mathrm{v+g}}$*-site on* $\widehat{V}(K)$, denoted $\widehat{V}_{\widehat{\mathrm{v+g}}}$, is the category $\mathrm{Op}(\widehat{V}_{\widehat{\mathrm{v+g}}}(K))$ whose objects are of the form $\widehat{W}(K)$ with $W \in \mathrm{Op}(V_{\mathrm{v+g}})$, the morphisms are the inclusions and the admissible covers $\mathrm{cov}(\widehat{U}(K))$ of $\widehat{U}(K) \in \mathrm{Op}(\widehat{V}_{\widehat{\mathrm{v+g}}})$ are covers by objects of $\mathrm{Op}(\widehat{V}_{\widehat{\mathrm{v+g}}}(K))$ with finite subcovers.

Remarks

To be fully precise...

- one needs to define the $\mathrm{v+g}$-site on definable subsets of $V(K) \times \Gamma^n_{K,\infty}$, and also the $\widehat{\mathrm{v+g}}$-site on $\widehat{V}(K) \times \Gamma^n_{K,\infty}$.
- Before working in $\widehat{V}$, one first develops a sheaf cohomology theory for definable subsets of $\Gamma^n_{K,\infty}$. For Γ^n_K, this coincides with cohomology theories defined for o-minimal structures by Edmundo, Peatfield, Jones. But adding ∞ does change things...
- To some extend this can be seen as a “tropical cohomology theory” for (o-minimal expansions of) ordered divisible abelian groups with a point at infinity.

Remarks

The proofs for $\widehat{V}$ (quasi-projective) are achieved using

- The deformation retraction and the results over Γ_∞
- Showing that the deformation retraction is a morphism of sites
- Key results about definable normality and definably connected (hats of) definable sets (both in $\Gamma_{K,\infty}$ an $\widehat{V}(K)$).
- Plus a bunch of (co)homological diagrams :)

Final comments:

There are lots of questions naturally appearing in this context!

- Extra work is needed to get finiteness and invariance without supports.
- analogues of étale cohomology for $\widehat{V}(K)$?
- definably proper maps, (Verdier) duality, etc.
- define the corresponding fundamental group for $\widehat{V}(K)$, and show they have tame properties in this context.

Thanks for your attention

Let's talk about Berkovich spaces (just a bit more):

Let F be a complete rank 1 non-archimedean field not necessarily algebraically closed (but keep the additive notation). Let $F^{\max}$ be an algebraically closed maximally complete extension of F with value group $\mathbb{R}$.

- For every $V \in \mathrm{Var}_K$, there is a space of types $B_F(V)$ (in ACVF) with a natural topology such that $B_F(V)$ is homeomorphic to V^{an}.
- When F is algebraically closed (non-trivially valued), then $B_F(V)$ sits in between $\widehat{V}(F)$ and $\widehat{V}(F^{\max})$.
- More generally, there is a continuous surjective map

$$\pi \colon \widehat{V}(F^{\max}) \to B_F(V)$$

- $\widehat{V}(F)$ is definably compact $\Leftrightarrow$ $\widehat{V}(F^{\max})$ is compact $\Leftrightarrow$ $B_F(V)$ is compact.

Using π, one can recover results on sheaf cohomology for $B_F(V)$ (and hence for V^{an}) using our results on the cohomology of $\widehat{V}(F^{\max})$.