Group compactifications in continuous logic
with applications to multiplicative combinatorics

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Outline

(1) Compactifications of pseudofinite and pseudo-amenable groups
arXiv 2023, joint with Hrushovski and Pillay

(2) An analytic version of stable arithmetic regularity
arXiv 2024, joint with Pillay

Part 1

Compactifications of pseudofinite and pseudo-amenable groups

Original motivation

Question (Zilber)

Is there a pseudofinite group which admits a surjective homomorphism to a compact simple Lie group?

In 2017, Pillay established a negative answer under the additional assumption that the homomorphism is “internal”.

The proof used nilpotency of Hrushovski’s “Lie model” for a pseudofinite approximate group, proved by Breuillard, Green, and Tao.

Shortly after, Nikolov, Schneider, and Thom gave a full negative answer. Their proof uses the classification of finite simple groups.

Pillay's theorem

Let M be a first order-structure.

A map $f\colon M \to X$, with X a compact Hausdorff space, is called definable if the preimages of two disjoint closed sets in X can be separated by a definable set in M.

Equivalently, f extends to a (necessarily unique) continuous function on $S_1(M)$.

Let G be a group. A (group) compactification of G is a homomorphism $\pi\colon G \to K$ with dense image, where K is a compact Hausdorff group.

Theorem (Pillay 2017)

Suppose G is a pseudofinite structure expanding a group, and $\pi\colon G \to K$ is a definable compactification. Then K^0 is abelian.

Applications

Additive combinatorics: Given a finite **abelian** group G, study the relationship between the algebraic structure of a set $A \subseteq G$ and the dynamics of the sum/difference sets $A + A$, $A - A$, etc.

Sets with "good algebraic structure" in abelian groups include subgroups and Bohr neighborhoods.

A $(\delta, \mathbb{T}^n)$-Bohr neighborhood in a group G is a subset of the form $\tau^{-1}(U)$ where $\tau\colon G \to \mathbb{T}^n$ is a homomorphism and $U \subseteq \mathbb{T}^n$ is the open identity neighborhood of radius δ.

Pillay's result allows for the use of Bohr neighborhoods in the noncommutative setting (multiplicative combinatorics) as well.

- (C., Pillay, Terry 2018) Structure of "VC sets" in finite groups.
- (C. 2018) Bogolyubov's Lemma for arbitrary finite groups.

Applications

Let G be a pseudofinite structure expanding a group, and let $\pi\colon G \to K$ be a definable compactification.

In practice, the Peter-Weyl Theorem can be used to assume K is compact Lie group. By Pillay's result, K^0 is a compact, connected, abelian Lie group, hence $K^0 \cong \mathbb{T}^n$ for some n.

$H := \pi^{-1}(K^0)$ is a definable finite-index subgroup. So if $U \subseteq K^0$ is an identity neighborhood, then $\pi^{-1}(U)$ is a Bohr neighborhood in H. However, $\pi^{-1}(U)$ is not necessarily definable.

In the previous applications, this is addressed with an elaborate process involving definable "approximate Bohr sets", and some nontrivial compactness arguments.

A “simpler” proof of Pillay’s result

Let G be a pseudofinite structure expanding a group and let $\pi \colon G \to K$ be a definable compactification. We want to prove K^0 is abelian.

By Peter-Weyl, we may assume K is a closed subgroup of a unitary group $U(n)$. Let d be the metric induced by the operator norm.

A map f from a group H to K is an ϵ-approximate homomorphism if $d(f(xy), f(x)f(y)) < \epsilon$ for all $x, y \in H$.

Claim. For any $\epsilon > 0$, there is a finite group H and an ϵ-approximate homomorphism $f \colon H \to K$ such that $f(H)$ is an ϵ-net in K.

Theorem (Turing 1938)

Any compact Lie group satisfying the conclusion of the Claim has an abelian connected component.

A rigidity theorem for approximate homomorphisms

Let K be a closed subgroup of a unitary group $U(n)$.

Theorem (Kazhdan 1982/ Alekseev-Glebskii-Gordon 2001)

Let H be a finite group. Suppose $f\colon H \to K$ is an ϵ-approximate homomorphism with ϵ sufficiently small (depending on K). Then there is a homomorphism $\tau\colon H \to K$ with $d(\tau(x), f(x)) < 2\epsilon$ for all $x \in H$.

A "modern" proof of Turing's result

Let K be a closed subgroup of a unitary group $U(n)$.

Suppose for all $\epsilon > 0$ there is an ϵ-approximate homomorphism from a finite group to K whose image is an ϵ-net. We'll show K^0 is abelian.

By Kazhdan/AGG, for any $m > 0$, there is a finite subgroup $H_m \leq K$, which is a $\frac{1}{m}$-net.

Let $\mathcal{U}$ be a nonprincipal ultrafilter. Then $\lim_{\mathcal{U}} \colon \prod_{\mathcal{U}} H_m \to K$ is a surjective group homomorphism.

Jordan's Theorem (1878) Any finite subgroup of $GL(n)$ has an abelian subgroup of index at most $d = O_n(1)$.

Fix $A_m \leq H_m$ abelian of index at most d. Then $A = \lim_{\mathcal{U}}(\prod_{\mathcal{U}} A_m)$ is an abelian subgroup of K of index at most d. The closure of A is abelian and contains K^0.

Aftermath

Schur extended Jordan's Theorem to torsion subgroups of $GL(n)$.

Kazhdan/AGG holds for approximate homomorphisms from amenable groups to compact Lie groups.

Corollary (CHP 2023)

Suppose G is elementarily equivalent to an ultraproduct of amenable torsion groups. Then any definable compactification of G has an abelian connected component.

Question: Does this hold for abstract compactifications?

The next steps

The previously mentioned applications of Pillay's theorem in multiplicative combinatorics involve the following process.

- ∗ Replace a definable compactification of a pseudofinite group by a definable ϵ-approximate homomorphism with finite image.
- ∗ Prove a desired result in terms of “approximate Bohr sets”.
- ∗ Recover genuine Bohr sets via Kazhdan/AGG.

While effective, this process was cumbersome, technical, and non-uniform.

Building Kazhdan/AGG into the logic

Let $\mathcal{L}$ be a language containing the language of groups.

Let $G = \prod_{\mathcal{U}} G_n$ where each G_n is an $\mathcal{L}$-structure expanding an amenable group.

Let K be a compact Lie group.

Theorem (CHP 2023; from Kazhdan/AGG)

Suppose $\tau\colon G \to K$ is a definable homomorphism. Then there are homomorphisms $\tau_n\colon G_n \to K$ such that $\tau = \lim_{\mathcal{U}} \tau_n$.

View (G, K, τ) as a two-sorted structure in continuous logic.

Corollary. (G, K, τ) is canonically isomorphic to $\prod_{\mathcal{U}}(G_n, K, \tau_n)$.

Now suppose $\mathcal{L}$ is a continuous language. Then the Theorem is still true, but the Corollary requires a modulus of uniform continuity for τ.

Ex. If each G_n has a discrete metric, then this is not an issue.

Part 2

An analytic version of stable arithmetic regularity

Stability for relations and functions

Let V be a set.

A binary relation E on V is k-stable if there do not exist $a_1, \ldots, a_k, b_1, \ldots, b_k \in V$ such that $E(a_i, b_j)$ holds if and only if $i \leq j$.

Fix $\boldsymbol{k} \colon \mathbb{R}^+ \to \mathbb{Z}^+$. Then $f \colon V \times V \to [\text{-}1, 1]$ is $\boldsymbol{k}$-stable if for all $\epsilon > 0$, there do not exist $a_1, \ldots, a_{\boldsymbol{k}(\epsilon)}, b_1, \ldots, b_{\boldsymbol{k}(\epsilon)} \in V$ such that for all $i < j$,

$$|f(a_i, b_j) - f(a_j, b_i)| \geq \epsilon.$$

Stability of E is equivalent to stability of $\mathbf{1}_E$, up to a uniform change k.

Stable regularity

Malliaris-Shelah (2011) Given a stable relation E on a finite set V, there is a partition $\{V_i\}$ of bounded size such that E is almost complete or almost empty on each pair $V_i \times V_j$.

Chavarria-C.-Pillay (2021) Given a stable function f on a finite set V, there is a partition $\{V_i\}$ of bounded size such that f is almost constant on each pair $V_i \times V_j$.

The proof of CCP uses "local stability" in continuous logic, and does not provide quantitative bounds.

Stable arithmetic regularity

A subset A of a group G is k-stable if the relation $xy \in A$ is k-stable.

Theorem

Fix $\epsilon > 0$ and $k \geq 1$. Then for any finite group G and k-stable set $A \subseteq G$, there is a subgroup $H \leq G$ of index $O_{k,\epsilon}(1)$ such that for any left coset C of H, $|A \cap C| < \epsilon|H|$ or $|A \cap C| > (1-\epsilon)|H|$.

(Terry & Wolf 2017) $G = \mathbb{F}_p^n$ and $O_{k,\epsilon}(1) = p^{(1/\epsilon)^{O_k(1)}}$

(C., Pillay, & Terry 2017) G arbitrary and $O_{k,\epsilon}(1)$ ineffective

(Terry & Wolf 2018) G abelian and $O_{k,\epsilon} = 2^{(1/\epsilon)^{O_k(1)}}$

(C. 2020) G arbitrary and $O_{k,\epsilon} = (1/\epsilon)^{O_k(1)}$

Stable arithmetic regularity for functions

If G is a group then $f\colon G \to [\text{-}1, 1]$ is ***k*-stable** if $f(x \cdot y)$ is ***k*-stable.**

Following the analogy, one might guess that if $f\colon G \to [\text{-}1, 1]$ is a stable function on a finite group, then there is subgroup $H \leq G$ of bounded index such that f is almost constant on each left coset of H.

But this is not the case.

Stability of the inner product

Fact

There is $\boldsymbol{k}\colon \mathbb{R}^+ \to \mathbb{Z}^+$ such that for any Hilbert space, the inner product is $\boldsymbol{k}$-stable when restricted to the closed unit ball.

This is immediate from stability of Hilbert spaces in continuous logic, but also follows from earlier literature in functional analysis (e.g., Grothendieck, Krivine & Maurey).

Example

Let $G = \mathbb{Z}/p\mathbb{Z}$ where $p > 2$ is prime and let $A = \{0, 1, \ldots, \frac{p}{2}\}$. Define $f\colon G \to [0, 1]$ so that $f(x) = |A \cap (x + A)|/|G|$. Then f is $\boldsymbol{k}$-stable for $\boldsymbol{k}$ as in the Fact. But f takes values arbitrarily close to 0 and $\frac{1}{2}$ as $p \to \infty$.

Bohr neighborhoods

To get a correct regularity result for stable functions on finite groups, we need to replace subgroups by Bohr neighborhoods.

We will also expand our focus to amenable groups.

Definition

Let K be a compact metrizable group, and let G be an arbitrary group. Then a (δ, K)-Bohr neighborhood in G is a subset of the form $\tau^{-1}(U)$ where $\tau\colon G \to K$ is a homomorphism and $U \subseteq K$ is the open identity neighborhood of radius δ.

In the amenable setting we focus on $(\delta, U(n))$-Bohr neighborhoods.

Main Result

Let G be an amenable group with a fixed left-invariant finitely additive probability measure μ. Fix $f\colon G \to [\text{-}1, 1]$ and $\epsilon, \zeta > 0$.

We say f is ζ-almost ϵ-constant on $B \subseteq G$ if there is some $B' \subseteq B$ such that $\mu(B') \geq (1-\zeta)\mu(B)$ and $|f(x) - f(y)| \leq \epsilon$ for all $x, y \in B'$.

Theorem (C.-Pillay 2024)

Assume $f\colon G \to [\text{-}1, 1]$ is $\boldsymbol{k}$-stable. Then there is a $(\delta, U(n))$-Bohr neighborhood $B \subseteq G$, with $\delta^{-1}, n \leq O_{\boldsymbol{k},\epsilon,\zeta}(1)$, such that f is ζ-almost ϵ-constant on all translates of B.

- If G is abelian, then B is a $(\delta, \mathbb{T}^n)$-Bohr neighborhood.
- If G is finite, then one may assume B is $(\delta, \mathbb{T}^n)$-Bohr neighborhood in a normal subgroup of index $O_{\boldsymbol{k},\epsilon,\zeta}(1)$.
- ζ can be replaced by $\zeta(\delta, n)$ where $\zeta\colon \mathbb{R}^+ \times \mathbb{Z}^+ \to \mathbb{R}^+$ is fixed.

Proof ingredients

Suppose not. We obtain counterexamples (G_s, f_s) to $s = O_{\boldsymbol{k},\epsilon,\zeta}(1)$. So G_s is an amenable group and $f_s \colon G_s \to [\text{-}1, 1]$ is $\boldsymbol{k}$-stable.

View (G_s, f_s) as a metric structure with a discrete metric in the language of groups expanded by a $[\text{-}1, 1]$-valued unary predicate.

Key point: The "d-metric" on φ-types is not necessarily discrete.

Let $(G, f) = \prod_{\mathcal{U}}(G_s, f_s)$ for some nonprincipal ultrafilter.

Each G_s comes with a left-invariant measure μ_s, which we view as a linear functional. We get a linear functional $\mu = \lim_{\mathcal{U}} \mu_s$, viewed as a Keisler measure (a regular Borel probability measure on $S_1(G)$).

Proof ingredients

Let $(G, f) = \prod_{\mathcal{U}}(G_s, f_s)$ and μ be as before.

Theorem (CP)

Fix $\epsilon > 0$. Then there are $n \geq 0$, $\delta > 0$, and a definable homomorphism $\tau \colon G \to U(n)$ such that if U is the open identity neighborhood of radius δ, then on each translate of $\tau^{-1}(U)$, f is ϵ-constant up a μ-null set.

The proof uses local stability, work of Ellis and Nerurkar (1982) on weakly almost periodic flows, and the Peter-Weyl Theorem (to get to unitary groups).

Now we use the previous machinery with Hrushovski to view $(G, f, U(n), \tau)$ as an ultraproduct $\prod_{\mathcal{U}}(G_s, f_s, U(n), \tau_s)$.

An application

Bogolyubov's Lemma (Ruzsa 1994)

Let G be a finite abelian group and fix $A \subseteq G$ with $|A| = \alpha|G|$ for some $\alpha > 0$. Then there is a $(\frac{1}{4}, \mathbb{T}^n)$-Bohr neighborhood $B \subseteq G$, with $n \leq \alpha^{-2}$, such that $B \subseteq 2(A - A)$.

Theorem

Let G be an amenable group with left-invariant measure μ and fix $A \subseteq G$ with $\mu(A) = \alpha > 0$. Then there is a $(\delta, U(n))$-Bohr neighborhood $B \subseteq G$, with $\delta^{-1}, n \leq O_\alpha(1)$, such that $B \subseteq (AA^{-1})^2$.

- (Beiglböck, Bergelson, Fish 2009) Given G countable amenable, and $A \subseteq G$ with $\mu(A) > 0$, there is some Bohr neighborhood B in $(AA^{-1})^2$.
- (C. 2018) Theorem for finite G, with a $(\delta, \mathbb{T}^n)$-Bohr neighborhood in a normal subgroup; using combinatorial tools of Sanders (2010).
- (C., Pillay, Hrushovski 2023) Theorem as stated; using a variation of the Stabilizer Theorem by Montenegro, Onshuus, Simon (2016).

Bogolyubov from analytic stable arithmetic regularity

Let G be an amenable group with left-invariant measure μ.

Given bounded functions $f, g\colon G \to \mathbb{R}$, define the convolution

$$(f * g)(x) = \int f(t)g(t^{-1}x)\,d\mu(t).$$

Then $f * g$ is $\boldsymbol{k}$-stable where $\boldsymbol{k}$ witnesses stability of inner products.

Fix $A \subseteq G$ with $\mu(A) = \alpha > 0$. Define $f\colon G \to [0,1]$ so that $f(x) = \mu(A \cap xA)$. Then $f = \mathbf{1}_A * \mathbf{1}_{A^{-1}}$, and hence is $\boldsymbol{k}$-stable.

Apply CP (2024) with $\epsilon > 0$ and $\zeta\colon \mathbb{R}^+ \times \mathbb{Z}^+ \to \mathbb{R}^+$ depending on α. This yields a $(\delta, U(n))$-Bohr neighborhood B such that $\delta^{-1}, n \leq O_\alpha(1)$ and f is $\zeta(\delta, n)$-almost ϵ-constant on all translates of B.

Using elementary covering arguments, and the choice of ϵ, ζ, one can show $B \subseteq (AA^{-1})^2$.

Future Directions

* C., Chavarria, Pillay (2021) applies to functions $f\colon V \times V \to [\text{-}1, 1]$ that are only (k, ϵ)-stable for a single k and ϵ. Is there an arithmetic analogue of this?

* Prove an analytic version of NIP arithmetic regularity.

* More applications of stable arithmetic regularity that exploit stability of convolutions.

thank you