

Zeroes and \mathbb{Q} -points of analytic or oscillatory functions

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- for $x = \frac{a}{b}, y = \frac{p}{q} \in \mathbb{Q}$ with $a \wedge b = p \wedge q = 1$,

$$\text{ht}(x, y) := \max\{|a|, |b|, |p|, |q|\},$$

- $f : [0, 1] \rightarrow \mathbb{R}$ (or $f : \bar{D}(0, 1) \rightarrow \mathbb{C}$) an analytic function on a neighbourhood of $[0, 1]$ (or $\bar{D}(0, 1)$),
- Γ_f the graph of f ,
- For $T \geq 1$

$$\Gamma_f(\mathbb{Q}, T) := \{(x, f(x)) \in \Gamma_f \cap \mathbb{Q}^2; \text{ht}(x, f(x)) \leq T\}.$$

- for $d \in \mathbb{N}$, $\mathcal{P}_d \subset \mathbb{K}[X, Y]$ the space of polynomials of degree $\leq d$, $\mathbb{K} = \mathbb{R}$ or \mathbb{C} .
- $Z_d(f) := \sup_{P \in \mathcal{P}_d \setminus \{0\}} \#\{P(z, f(z)) = 0\} \in \mathbb{N} \cup \{\infty\}$.

Remark. $Z_d(f)$ is the maximum number of intersection points between Γ_f and algebraic curves of degree $\leq d$.

Call a bound for $Z_d(f)$ a *Bézout bound*.

Remark. In case f is a polynomial, $Z_d(f)$ is polynomially bounded in d (and $\deg f$) when the intersection is transverse.

Remark. By the curve selection lemma

$$Z_d(f) < \infty \text{ or } \Gamma_f \text{ contains a semialgebraic curve (of dimension 1).}$$

We assume that for any $d \in \mathbb{N}$, $Z_d(f) < \infty$, that is f is a *transcendental function*.

Theorems: $\#\Gamma_f(\mathbb{Q}, T)$ is *sub-polynomial*.

- [Bombieri & Pila 1989]:

$$\forall \varepsilon > 0 \exists C_{f,\varepsilon} \text{ s.t. } \#\Gamma_f(\mathbb{Q}, T) \leq C_{f,\varepsilon} T^\varepsilon.$$

- [Pila & Wilkie 2006]: Same result for $X \subset \mathbb{R}^n$ an o-minimal set:

$$\forall \varepsilon > 0 \exists C_{X,\varepsilon} \text{ s.t. } \#X^{\text{trans}}(\mathbb{Q}, T) \leq C_{X,\varepsilon} T^\varepsilon.$$

$$X^{\text{trans}} := X \setminus X^{\text{alg}},$$

$$X^{\text{alg}} := \{x \in X; \exists S \text{ semialgebraic of pure dimension 1, s.t. } x \in S \subset X\}.$$

Remark. [Pila 2006, Prop. 2.4]: $\Gamma_f(\mathbb{Q}, T)$ is contained in a certain number of hypersurfaces of \mathbb{R}^2 of degree d , this number being bounded by

$$C_{f,d} T^{\varepsilon d},$$

- $C_{f,d}$ is polynomial in d : comes from $|f^{(p)}|$.
- for $d = \lfloor \log T \rfloor$, $T^{\varepsilon d}$ is a constant independent of T .

Consequence. For $f : [0, 1] \rightarrow \mathbb{R}$ analytic, transcendental, $\exists Q \in \mathbb{R}[X]$ s.t.

$$\#\Gamma_f(\mathbb{Q}, T) \leq Q(\log T) \times Z_{\lfloor \log T \rfloor}(f).$$

In particular when f has a polynomial Bézout bound

$$\exists \alpha, \beta \text{ s.t. } \#\Gamma_f(\mathbb{Q}, T) \leq \alpha \log^\beta T \quad (\text{vs } \forall \varepsilon > 0, \#\Gamma_f(\mathbb{Q}, T) \leq \alpha C_\varepsilon T^\varepsilon)$$

Natural questions arising.

1. *Compact interval.* Find analytic functions $f : [0, 1] \rightarrow \mathbb{R}$ with $Z_d(f)$ polynomially bounded in d .
2. *Non-compact interval.* Find \mathcal{C}^∞ a class of $f : [a, +\infty[\rightarrow \mathbb{R}$ with
 - $\forall p \geq 0$, suitable decay for $|f^{(p)}|$.
 - $\forall d \geq 1, \forall I \subset [a, +\infty[$, polynomially controlled $Z_d(f|_I)$ w.r.t. d and $\text{length}(I)$,

Bounding $\#\Gamma_f(\mathbb{Q}, T)$ for (oscillatory) functions $f : [a, +\infty[\rightarrow \mathbb{R}$ and more generally for (oscillatory) curves $\Gamma \subset \mathbb{R}^n$.

Definition. The C^∞ -parametrization $\gamma = (f, g) : [a, +\infty[\rightarrow \mathbb{R}^2$ of the curve $\Gamma \subset \mathbb{R}^2$ is *slow* when

- $\exists u \in \mathbb{R}, \forall x, |u - f(x)| \leq b(x) \searrow 0,$
- $\forall p \geq 0, \forall x, \left| \frac{f^{(p)}(x)}{p!} \right| \leq \varphi_p(x), \left| \frac{g^{(p)}(x)}{p!} \right| \leq \varphi_p(x),$ where

$$\exists \text{ constants } A, B, C, D \text{ s.t. } \forall x, \varphi_p(x) = D \left(A p^B \frac{\log^C x}{x} \right)^p.$$

Remark. Functions satisfying 2 yield a subalgebra of $C^\infty([a, +\infty[)$ stable under derivation.

Example. $g := h \circ \log^\ell$, where $\ell \geq 1$ and $\exists \alpha, \forall p \geq 0, |h^{(p)}(x)| \leq \alpha^p$.

Definition. $\varphi : [a, +\infty[\rightarrow \mathbb{R}$ is a *height control function* of γ when

$$\gamma^{-1}(\Gamma(\mathbb{Q}, T)) \subset [a, \varphi(T)].$$

Examples. For f slow:

- when $u \in \mathbb{Q}$ and f doesn't take the value u , one can take $\varphi(T) = b^{-1}\left(\frac{K}{T}\right)$, for some K .
- When $u \notin \mathbb{Q}$, and is not a U -number of degree 1 in Mahler's classification, one can take $\varphi(T) = b^{-1}\left(\frac{1}{TK}\right)$, for some K .

Theorem. [C. & Miller 2017] Let γ be a **slow** parametrization of a transcendental curve Γ , with height control function φ , $T \geq 1$, $d \geq 1$ and

$$\mathcal{B}_{d,A} := \sup_{P \in \mathcal{P}_d \setminus \{0\}} \#P^{-1}(0) \cap \gamma([a, A]),$$

then

$$\#\Gamma(\mathbb{Q}, T) \leq \alpha \log^\delta(T) \times \log^\nu(\varphi(T)) \times \mathcal{B}_{\log T, \varphi(T)}.$$

Consequence. When $e^{\varphi(T)}$ and $\mathcal{B}_{d,A}$ are polynomially bounded in T, d, A then

$$\#\Gamma(\mathbb{Q}, T) \leq \alpha \log^\beta(T).$$

Examples. Built on elementary functions composed with \log^ℓ
 \implies Slowness & $\mathcal{B}_{d,A}$ suitable ([Khovanskiĭ]).

- log-spirals: $(\frac{1}{x^F} \sin \circ \log^\ell, \frac{1}{x^G} \cos \circ \log^q)$, $F, G > 0$, $\ell, q \in \mathbb{N}^*$.
- $(\log 2 + \frac{\arctan \log^2 x}{x^5(2 + \cos^3 \log x)}, \pi + \frac{\sin \log^2 x}{\sqrt{x}(1 + \log \log x)})$, here $b(x) = \frac{1}{x^5}$,
 $\varphi(T) = T^{\frac{1}{5}}$
- Graphs: $x \mapsto \sin \log^\ell x$ max. sol. of Euler equation $x^2 y'' + x y' + y = 0$
 $(\sin \log^\ell x \text{ defines } \mathbb{Z} \text{ over } \mathbb{R} \iff \ell > 1)$.

Question. In general what's going on without o-minimality?

\mathcal{R} an expansion of \mathbb{R}

- In case \mathcal{R} defines \mathbb{Z} , no hope for interesting bounds for definable sets,
- In case \mathcal{R} does not define \mathbb{Z} and is not o-minimal:
 - $(\mathbb{R}, \bar{\mathbb{Q}})$ no sub-polynomial bound ($\bar{Q}^{trans} = \bar{Q}$),
 - $(\mathbb{R}, \{2^{\mathbb{Z}}\})$ sub-polynomial bound $E^{trans} = \emptyset!$

Related open questions.

1. Mahler's classification of \emptyset -definable points of (\mathbb{R}, f) ?
2. $\log^{\beta} T$ -bound for the expansion of \mathbb{R} by any log-spiral and by restricted \sin and \exp (cf [Binyamini & Novikov 2016])?
3. $\log^{\beta} T$ -bound for \emptyset -definable sets of the expansion of \mathbb{R} by any log-spiral? By $\sin \log$?
4. $\log^{\beta} T$ -bound for \emptyset -definable sets of $(\mathbb{R}, +, \cdot, E)$ where E boolean comb. of open sets s.t. $(\mathbb{R}, +, \cdot, E)$ does not define \mathbb{Z} ? etc.

$f : \bar{D}(0, 1) \rightarrow \mathbb{C}$ transcendental analytic function

How to prescribe polynomial bounds for $Z_d(f)$?

Remark. $Z_d(f) \leq K_d < \infty$ holds for f in any o-minimal structure. On the other hand $Z_d(f)$ may be polynomially bounded in d while f is not o-minimal (see [Gwoździewicz-Kurdyka-Parusiński 1999]).

Remark. Even when f is analytic, the asymptotic of $Z_d(f)$ is difficult to predict: for any $\zeta \in]0, 1[$, there exists $f : D \rightarrow \mathbb{C}$ analytic such that for a sequence of degrees d going to ∞ ,

$$Z_d(f) \geq e^{d^\zeta}.$$

(see [Surroca 2002, 2006],[Pila 2004])

Remark. For f entire of finite order $:= \limsup_{r \rightarrow \infty} \frac{\log \log \max_{D_r} |f|}{\log r} < \infty$, for a certain sequence of degrees going to ∞

$$Z_d(f) \leq Cd^2 \quad (\text{best possible asymptotic}).$$

What's known on the asymptotic of $Z_d(f)$?

- $f(z) = e^z$ has a polynomial Bézout bound:
[Tijdeman 1971],
- Elementary functions have polynomial Bézout bounds:
[Khovanskii 1991],
- Entire functions with $0 < \text{lower order} \leq \text{finite order} < \infty$, have polynomial Bézout bounds:
[Coman & Poletsky 2003, 2007],[Brudnyi 2008],[Boxall & Jones 2015].
- Specific functions like the Riemann ζ function, the Euler Γ function, have polynomial Bézout bounds:
[Coman & Poletsky 2007],[Masser 2011],[Besson 2011, 2014],[Boxall & Jones 2013].
- (Compact) solutions of some algebraic differential equations have polynomial Bézout bounds:
[Binyamini 2016].

Notation.

- $\Psi : (\mathbb{C}, 0) \rightarrow (\mathbb{C}^n, 0)$ analytic,
- $Q_1, \dots, Q_m : (\mathbb{C}^n, 0) \rightarrow (\mathbb{C}, 0)$ analytic maps.

For $\lambda = (\lambda_1, \dots, \lambda_m) \in \mathbb{C}^n$, let

- $Q_\lambda(z) = \sum_{i=1}^m \lambda_i Q_i(\Psi(z)) = \sum_{k=0}^{\infty} v_k(\lambda) z^k$, $v_k(\lambda)$ linear forms on \mathbb{C}^m .
- $L_i := \{v_0 = v_1 = \dots = v_i = 0\}$, $\mathbb{C}^m \supseteq L_0 \supseteq L_1 \supseteq \dots \supseteq L_i \supseteq \dots$

This sequence stabilizes at *the Bautin index* $b = b_{\Psi, Q_1, \dots, Q_m}$:

$$L_{b-1} \supsetneq L_b = L_{b+1} = \dots$$

Remark. $\lambda \in L_b \iff \forall k \geq 0, v_k(\lambda) = 0$.

Application.

- $n = 2$, $\Psi(z) = (z, f(z))$ $Q_i = X^j Y^p$, $j, p \in \llbracket 0, d \rrbracket$, $m = (d + 1)^2$,

$$\text{then } Q_\lambda(z) = \sum_{j=0}^d p_j(z) f^j(z), \text{ deg } p_j \leq d.$$

Remarks.

- The maximum number (w.r.t. λ) of zeroes of Q_λ bounds $Z_d(f)$.
- Since $\lambda \in L_b \iff \forall k \geq 0, v_k(\lambda) = 0$, when f is transcendental $\lambda \neq 0$ cannot cancel all v_k , therefore $L_b = \{0\}$.
- $\mathbb{C}^m \supseteq L_0 \supseteq \dots \supseteq L_b = \{0\}$,
therefore b minimal for $b = m - 1 = d^2 + 2d$.
- But b may be very large!

Remarks.

- For $d \geq 1$

$$b = \max_{P \in \mathcal{P}_d \setminus \{0\}} \#\{\text{mult}_0 P(z, f(z))\}$$

- $(b = b_d)_{d \geq 1}$ measures the transcendency of f : the faster $(b = b_d)_{d \geq 1}$ goes to ∞ the less f seems transcendental.
- $\lim_{r \rightarrow 0} \max_{P \in \mathcal{P}_d \setminus \{0\}} \#\{z \in D_r; P(z, f(z)) = 0\} \leq b$.

Notation.

- $a_i^j := \frac{1}{i!} (f^j)^{(i)}(0)$
- After reduction, the matrix with lines the v_k 's is:

$$M = \begin{pmatrix} a_{d+1}^1 & \text{---} & a_1^1 & \cdots & a_{d+1}^d & \text{---} & a_1^d \\ | & & | & & | & & | \\ a_b^1 & \text{---} & a_{b-d}^1 & \cdots & a_b^d & \text{---} & a_{b-d}^d \end{pmatrix} \quad (1)$$

- Δ the absolute value of a nonzero $(d^2 + d) \times (d^2 + d)$ minor of M .
(exists since $L_b = \{0\}$!)

Theorem. [C. & Yomdin 2016] On $D_{\frac{1}{4}}$:

$$Z_d(f) \leq 5b \log(4 + 2(b+1)) \frac{e^{2(d+1)^2 \log(d+1)}}{\Delta}.$$

Consequence. When there exist $R, S \in \mathbb{R}_+[X]$ s.t.

$$\forall d \in \mathbb{N}, \quad b \leq R(d) \quad \text{and} \quad \Delta \geq e^{-S(d)},$$

$Z_d(f)$ is polynomially bounded on $D_{\frac{1}{4}}$.

Notation. When $\forall k \geq 0$, $a_k = \frac{p_k}{q_k} \in \mathbb{Q}$ denote $h_\ell := \max\{|q_0|, \dots, |q_\ell|\}$.

Proposition 1. [C. & Yomdin 2016] For $f \in \mathbb{Q}\{z\}$, if there exist $R, S \in \mathbb{R}[X]$ s.t.

$$b_d \leq R(d) \quad \text{and} \quad h_\ell \leq e^{S(\ell)}$$

then $Z_d(f)$ is polynomially bounded.

Definition. f is *hypertranscendental* when f satisfies no algebraic differential equation.

Notation. For f hypertranscendental,

$$\eta_d := \max_{P \in \mathbb{Z}_d[X_0, \dots, X_d] \setminus \{0\}} \{\text{mult}_0 P(z, f(z), f'(z), \dots, f^{(d)}(z))\}$$

Proposition 2. For $f \in \mathbb{Q}\{z\}$, if there exist $R, S \in \mathbb{R}[X]$ s.t.

$$\eta_d \leq R(d) \quad \text{and} \quad h_\ell \leq e^{S(\ell)}$$

then f has a polynomial Bézout bound.

Theorem 1. [C. & Yomdin 2016] Assume $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k} \in \mathbb{C}\{z\} \setminus \mathbb{C}[z]$, with $n_k^2 \leq n_{k+1} \leq n_k^q$, for some $q > 2$, then

- f is hypertranscendental ([Ostrowski 1920]).
- $b_d \leq d^{q^2}$.

If furthermore $|a_k| \geq e^{-n_k^p}$, for some $p > 0$, then

- $Z_d(f) \leq 10(2d)^{q^2} (1 + qd^2 + 5d^{pq+3})$.

Theorem 2. (cf also [Binyamini 2016]) Assume $f(z)$ is a solution of a linear differential equation with coefficients in $\mathbb{Q}[z]$ and rational initial conditions. Then f has a polynomial Bézout bound.

Proof. b_d is polynomially bounded ([Nesterenko 88] or [Gabrielov 99]). The Taylor coefficients of f are rational and $h_\ell \leq e^{S(\ell)}$.

Notation.

- $[0, 1] = I \leftarrow \dots \leftarrow I^n \leftarrow I^{n+1} \leftarrow \dots \leftarrow I^\infty = \varprojlim_{n \in \mathbb{N}} I^n$
- $f = \sum_{k=0}^{\infty} a_k z^k \sim (a_k)_{k \in \mathbb{N}} \in I^\infty$
- μ the measure on I^∞ induced by cylinders $\pi_n^{-1}(G), G \in I^n$
 μ_n -measurable, where $\mu(\pi_n^{-1}(G)) := \mu_n(G)$.

Theorem 3. For μ -a.e. $f \in I^\infty$, $\exists U \in \mathbb{R}[X], Z_d(f) \leq U(d)$.

Part II

Random series: Proof

Let Δ_d be the determinant of the upper $d^2 + d$ matrix of M

$$\Delta_d = \begin{vmatrix} a_{d+1}^1 & \cdots & a_1^1 & \cdots & a_{d+1}^d & \cdots & a_1^d \\ \vdots & & \vdots & & \vdots & & \vdots \\ a_{d^2+2d}^1 & \cdots & a_{d^2+2d}^1 & \cdots & a_{d^2+2d}^d & \cdots & a_{d^2+2d}^d \end{vmatrix} \quad (2)$$

When $\Delta_d \neq 0$, b is minimal and polynomial = $d^2 + 2d$.

Δ_d is consider as a polynomial in $f_{|I^{d^2+d}}$ with arity $m_d := d^2 + d$ and

degree $n_d : \frac{d(d+1)}{2}$.

Proposition. $\forall p \in]0, 1[, \exists E_p \subset I^\infty, \mu(E_p) = p$ s.t.

$f \in E_p \implies \forall d, |\Delta_d(f)| \geq e^{-(\gamma_p+3)d^5}$ (with $\gamma_p \underset{p \rightarrow 1}{\sim} \log \frac{1}{1-p} \xrightarrow{p \rightarrow 1} +\infty$).

Proof of the Theorem.

- Since for $f \in E_p, \Delta_d(f) \neq 0, b_d = d^2 + 2d$.
- Since $|\Delta_d(f)| \geq e^{S(d)}$, there exists $T \in \mathbb{R}[X]$, s.t. $Z_d(f) \leq T(d)$.
- $\mu\left(\bigcup_{q \in \mathbb{N}^*} E_{1-\frac{1}{q}}\right) = 1$.

Proposition. $\forall p \in]0, 1[, \exists E_p \subset I^\infty, \mu(E_p) = p$ s.t.

$$f \in E_p \implies \forall d, |\Delta_d(f)| \geq e^{-(\gamma_p+3)d^5}.$$

Proof of Theorem. Fix $p \in]0, 1[$.

- For $d \geq \mathbb{N}^*$, denote $\theta_d := \frac{1-p}{\zeta(2)d^2}$.
- $V_d := \{f \in I^\infty; |\Delta_d(f)| \leq \varepsilon_d\}$, where ε_d is maximal and s.t. $\mu(V_d) = \theta_d$.
- $E_p := I^\infty \setminus \cup_d V_d$, $\mu(E_p) \geq 1 - (1-p) = p$, and $f \in E_p \implies \forall d, |\Delta_d(f)| \geq \varepsilon_d$.
- Now let us estimate ε_d .
- By [Remez inequality](#) and since $1 \leq |\Delta_d(f)|$ for specific f :

$$1 \leq |\Delta_d|_{I^{m_d}} \leq \left(\frac{4m_d}{\mu_{m_d}(V_d)}\right)^{n_d} |\Delta_d|_{V_d} \leq \left(\frac{4m_d}{\theta_d}\right)^{n_d} \varepsilon_d$$

- It finally gives for $f \in E_p$:

$$\varepsilon_d \geq \left(\frac{\theta_d}{4m_d}\right)^{n_d} \geq \dots \geq e^{-(\gamma_p+3)d^5}.$$

\mathbb{Q} -Theorem. For $f(z) = \sum_{k=0}^{\infty} a_k z^k$, assume one of the following conditions

1. $f \in \mathbb{Q}\{z\}$, $\exists R, S \in \mathbb{R}[X]$ s.t. $b_d \leq R(d)$, $h_l \leq e^{S(l)}$,
2. $f \in \mathbb{Q}\{z\}$, $\exists R, S \in \mathbb{R}[X]$ s.t. $\eta_d \leq R(d)$, $h_l \leq e^{S(l)}$,
3. $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k} \in \mathbb{R}\{z\}$, $n_k^2 < n_{k+1} \leq n_k^q$, for $q > 2$, $|a_k| \geq e^{-n_k^p}$, for $p > 0$,
4. f is a solution of a linear differential equation with coefficients in $\mathbb{Q}[z]$ with rational initial conditions,
5. f is a random series,

then there exists $\alpha, \beta > 0$, s.t. $\#\Gamma_f(\mathbb{Q}, T) \leq \alpha \log^\beta T$.

Remarks.

- Statement 4 has been obtained in [Binyamini 2016].
- Statement 5 is a consequence of [Boxall & Jones 2013]: a_0 transcendent with ‘good’ transcendency measure, such as S -numbers in Mahler’s classification (a.e. numbers are S -numbers) is enough for $\#\Gamma_f(\mathbb{Q}, T) \leq \alpha \log^\beta T$.

\mathbb{Q} -Theorem. For $f(z) = \sum_{k=0}^{\infty} a_k z^k$, assume one of the following conditions

1. $f \in \mathbb{Q}\{z\}$, $\exists R, S \in \mathbb{R}[X]$ s.t. $b_d \leq R(d)$, $h_l \leq e^{S(l)}$,
2. $f \in \mathbb{Q}\{z\}$, $\exists R, S \in \mathbb{R}[X]$ s.t. $\eta_d \leq R(d)$, $h_l \leq e^{S(l)}$,
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then there exists $\alpha, \beta > 0$, s.t. $\#\Gamma_f(\mathbb{Q}, T) \leq \alpha \log^{\beta} T$.

Remark (Siegel-Shidlovskii's theorem). A combination of comparable - but finer - conditions on f than 1, 2, 4, generalizing Lindemann-Weierstrass theorem (f E -function), gives

$$z \neq 0, z \in \bar{\mathbb{Q}} \implies f(z) \text{ transcendental.}$$

In particular $\#\Gamma_f(\mathbb{Q}, T) \leq 1$.