Some applications of model theory to geometric Ramsey theory

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Szemerédi regularity lemma

Theorem

[E. Szemerédi, 1975] If $\varepsilon > 0$*, then there exists* $K = K(\varepsilon)$ *such that: for any finite bipartite graph* $R \subseteq A \times B$*, there exist partitions* $A = A_0 \cup \ldots \cup A_k$ *and* $B = B_0 \cup \ldots \cup B_k$ *into non-empty sets, and a set* $\Sigma \subseteq \{1, \ldots, k\} \times \{1, \ldots, k\}$ *with the following properties.*

1. *Bounded size of the partition:* $k \leq K$.
2. *Few exceptions:* $\left|\bigcup_{(i,j)\in\Sigma} A_i \times B_j\right| \geq (1-\varepsilon)|A \times B|$.
3. ε*-regularity: for all* $(i,j) \in \Sigma$*, and all* $A' \subseteq A_i$, $B' \subseteq B_j$*, one has*
$$\left|\frac{|R \cap (A' \times B')|}{|A' \times B'|} - \frac{|R \cap (A_i \times B_j)|}{|A_i \times B_j|}\right| \leq \varepsilon.$$

Szemerédi regularity lemma: bounds and applications

- Has many applications in extreme graph combinatorics, additive number theory, computer science, etc.
- Exist various versions for weaker and stronger partitions, for hypergraphs, etc.
- Limitations:
  - [T. Gowers, 1997] The size of the partition $K(\varepsilon)$ grows as a tower of twos $2^{2^{\cdots}}$ of height $\left(1/\varepsilon^{16}\right)$.
  - Not so useful for sparse graphs.
- Can one obtain stronger versions for restricted families of graphs?

Stronger regularity for restricted families of graphs

1. [T. Tao, 2012] Algebraic graphs of bounded complexity in large finite fields (pieces of the partition are algebraic, no exceptional pairs, stronger regularity).
2. [L. Lovász, B. Szegedi, 2010] Graphs of bounded VC-dimension, i.e. NIP graphs (density arbitrarily close to 0 or 1, the size of the partition is bounded by a polynomial in $(\frac{1}{\varepsilon})$).
   2.1 [M. Malliaris, S. Shelah, 2011]: graphs without arbitrary large half-graphs, i.e. stable graphs (no exceptional pairs).
   2.2 Alon, Conlon, Fox, Gromov, Naor, Pach, Pinchasi, Radoičić, Sharir, Sudakov, Lafforgue, Suk: semialgebraic graphs of bounded complexity.

- All these cases are orthogonal to each other, and curiously have something to do with model theoretic classification theory.

Semialgebraic graphs

- A set $A \subseteq \mathbb{R}^d$ is *semialgebraic* if it is defined by a finite boolean combination of polynomial equalities and inequalities.
- We say that the *description complexity* of a semialgebraic set $A \subseteq \mathbb{R}^d$ is $\leq t$ if $d \leq t$ and A can be defined by a boolean combination of at most t polynomials, each of degree at most t.
- We say that a graph $R \subseteq \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ is semialgebraic if its edge relation is.
- Examples of semialgebraic graphs of bounded complexity: the incidence relation between points and lines on the plane, two parametrized families of semialgebraic varieties having a non-empty intersection, etc.

Semialgebraic Ramsey, 1

- We say that a pair of sets (A, B) is *R-homogeneous* if either $A \times B \subseteq R$ or $(A \times B) \cap R = \emptyset$.
- [N. Alon, J. Pach, R. Pinchasi, R. Radoičić, M. Sharir, "Crossing patterns of semi-algebraic sets", 1995]:

Theorem

For every $t \in \mathbb{N}$ there is some $\varepsilon > 0$ such that: if $R \subseteq \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$ is semialgebraic, of complexity bounded by t, then for any finite sets $A_i \subseteq \mathbb{R}^{d_i}$ there are some $A_i' \subseteq A_i$ such that $|A_i'| \geq \varepsilon |A_i|$ and (A_1', A_2') is R-homogeneous.
Moreover, $A_i' = A_i \cap S_i$, where S_i is a certain semialgebraic relation of complexity bounded in terms of t.

- Using this [J.Fox, M. Gromov, V. Lafforgue, A. Naor, and J. Pach, "Overlap properties of geometric expanders", 2010] obtain a semialgebraic regularity lemma — we'll return to it soon.

Semialgebraic Ramsey, 2

- By Tarski's quantifier elimination for real closed fields, this can be reformulated by saying that $(\mathbb{R}, +, \times)$ satisfies the following property.
- $(*)$ For every formula $\phi(x_1, x_2, z)$ there is some $\varepsilon > 0$ such that: for every choice of the parameter $c \in M^{|z|}$, for every finite $A_i \subseteq M^{|x_i|}$ there are some $A'_i \subseteq A_i$ such that $|A'_i| \geq \varepsilon |A_i|$ and (A'_1, A'_2) is $\phi(x_1, x_2, c)$-homogeneous. Moreover, $A'_i = A_i \cap S_i$, where $S_i \subseteq M^{|x_i|}$ is definable by a certain formula depending just on ϕ.
- $(*)$ is a property of $\mathrm{Th}(M)$: if it holds in one structure, then it holds in all structures elementarily equivalent to it.
- Which other theories satisfy $(*)$?

NIP theories

- Were introduced by [S.Shelah] for purposes of his classification theory: in some model M, some formula picks out all subsets of an infinite set.
- There is a rather elaborate theory of NIP theories based on invariant types, Keisler measures, indiscernible sequences, forking, etc — methods from infinitary combinatorics, ultrafilters, etc. Attracted a lot of attention recently.
- [C. Laskowski]: connection to finite VC-dimension, a notion from combinatorics introduced around the same time (central in computational learning theory), i.e. a theory is NIP iff all families of uniformly definable sets have finite VC-dimension.
- Key examples of NIP theories: algebraically closed fields, o-minimal theories (e.g. reals with exponentiation), p-adics, $ACVF$.

(*) implies NIP

- It follows from an easy probabilistic argument due to Pach that $(*)$ implies NIP (even without requiring definability of the homogeneous subsets).
- [S. Basu, 2007] Topologically closed graphs in *o*-minimal expansions of real closed fields satisfy $(*)$.
- Do all NIP theories satisfy $(*)$?
- No!

(*) fails in ACF_p

- For a finite field $\mathbb{F}_q$, let P_q be the set of all points in $\mathbb{F}_q^2$ and let L_q be the set of all lines in $\mathbb{F}_q^2$. Then $|P_q| = q^2$ and $|L_q| = q^2 + q \sim q^2$.
- Let $I \subseteq P_q \times L_q$ be the incidence relation. Using that fact that the lazy Szemerédi-Trotter bound $|I(P_q, L_q)| \leq |L_q| |P_q|^{\frac{1}{2}} + |P|$ is optimal in finite fields one can show:
- **Claim**. For any fixed $\delta > 0$, for all large enough q if $L_0 \subseteq L_q$ and $P_0 \subseteq P_q$ with $|P_0| \geq \delta q^2$ and $|L_0| \geq \delta q^2$ then $I(P_0, L_0) \neq \emptyset$.
- As every field of char p can be embedded into $\bar{\mathbb{F}}_p$, it follows that $(*)$ fails in $\bar{\mathbb{F}}_p$ (even without requiring definability of the homogeneous pieces) for I the incidence relation.

Results

- ACF_p is a nice stable theory. Turns out that stability is *the* problem.
- We will generalize $(*)$ (and further theory) in two directions: proving it for a larger class of theories (covering all *o*-minimal theories and *p*-adics) and for a larger class of measures (rather than just the counting ones, covering Lebesgue and Haar measures). Moreover, we will show that $(*)$ is equivalent to distality.
- Let us describe the context first.

Distal theories

- The class of *distal theories* was introduced by [P. Simon, 2011] in an attempt to capture the class of purely unstable NIP theories.
- The original definition is in terms of a certain property of indiscernible sequences (see later).

Theorem

[Ch., Simon, 2012] An NIP *theory T is distal if and only if for every formula $\phi(x, y)$ there is a formula $\psi(x, y_1, \ldots, y_n)$ such that for every $a \in M^{|x|}$ and every finite set $B \subset M^{|y|}$ there is some $c \in B^n$ such that $M \models \psi(a, c)$ and $\psi(x, c) \vdash \mathrm{tp}_{\phi}(a/B)$.*

- The proof uses some model theory along with some deep combinatorial results due to [J. Matoušek] and [N. Alon, D. Kleitman].
- It is enough to verify this property for formulas with $|x| = 1$.
- All *o*-minimal theories and $(\mathbb{Q}_p, +, \times)$ are distal.
- In a distal theory, any generically stable type is algebraic. So any distal theory is unstable, and ACVF is not distal.

Example: o-minimal theories are distal

- Let M be o-minimal and let $\phi(x, \bar{y})$ be given.
- For any $\bar{b} \in M^{|\bar{y}|}$, $\phi(x, \bar{b})$ is a finite union of intervals whose endpoints are of the form $f_i(\bar{b})$ for some definable $f_0(\bar{y}), \ldots, f_k(\bar{y})$.
- Given a finite set $B \subseteq M^{|\bar{y}|}$, the set of points $\{f_i(\bar{b}) : i < k, \bar{b} \in B\}$ divides M into finitely many intervals, and any two points in the same interval have the same ϕ-type over B.
- Thus, for any $a \in M$, either $a = f_i(\bar{b})$ for some $i < k$ and $\bar{b} \in B$, or $f_i(\bar{b}) < x < f_j(\bar{b}') \vdash \mathrm{tp}_\phi(a/B)$ for some $i, j < k$ and $\bar{b}, \bar{b}' \in B$.

Keisler measures

- A *(Keisler) measure* μ over a set of parameters $A \subseteq \mathbb{M}$ is a finitely additive probability measure on the boolean algebra $\mathrm{Def}_x(A)$ of A-definable subsets of $\mathbb{M}$.
- Every measure can be viewed as a measure defined on all clopen subsets of the compact space of types $S_x(A)$, and then it admits a unique extension to a regular Borel probability measure on $S_x(A)$.
- Let $\mathfrak{M}_x(A)$ be the space of measures over A. It can be naturally viewed as a closed subset of $[0,1]^{L(A)}$ with the product topology, so $\mathfrak{M}_x(A)$ is compact. Every type with a zero-one measure concentrated on it, thus $S_x(A)$ is a closed subset of $\mathfrak{M}_x(A)$.
- A global measure is a measure over $\mathbb{M}$.

Generically stable measures, 1

- A global measure μ is *smooth* over a small model $M \preceq \mathbb{M}$ if it is the unique measure extending $\mu|_M$.
- A global measure μ in an NIP theory is *generically stable* over a small model M if it is the unique $\mathrm{Aut}(\mathbb{M}/M)$-invariant Keisler measure extending $\mu|_M$.
- [Vapnik–Chervonenkis, 1971]+[E. Hrushovski, A. Pillay, P. Simon, 2010]. Generically stable measures in NIP theories are uniformly approximable by frequency measures: for every $\phi(x, y) \in L$ and $\varepsilon > 0$ there is some $n \in \mathbb{N}$ such that for every global generically stable measure μ there are some $a_0, \ldots, a_{n-1} \in \mathbb{M}$ such that for any $b \in \mathbb{M}$ we have $\left| \mu(\phi(x, b)) - \frac{|\{i<n : \models \phi(a_i, b)\}|}{n} \right| \leq \varepsilon$.

Generically stable measures, 2

- [Simon] A theory is distal iff every generically stable measure is smooth.
- Examples:
  - A global type viewed as a measure is smooth if and only if it is realized.
  - A counting measure concentrated on a finite set is smooth (in any theory).
  - Lebesgue measure on $[0, 1]$ (over reals, restricted to the definable sets) is smooth.
  - Haar measure on a ball over p-adics is smooth.
  - Let G be a definably compact group in an o-minimal theory. Then it admits a unique G-invariant measure, which is moreover smooth.

Generically stable measures, 3

- Given measures μ_1 on $\mathbb{M}^{|x_1|}$ and μ_2 on $\mathbb{M}^{|x_2|}$, we say that a measure μ on $\mathbb{M}^{|x_1|+|x_2|}$ is *a product measure* of μ_1 and μ_2 if for every definable set $S \subseteq \mathbb{M}^{|x_1|+|x_2|}$ such that $S = S_1 \times S_2$ with $S_i \subseteq \mathbb{M}^{|x_i|}$ definable, we have $\mu(S) = \mu_1(S_1)\,\mu_2(S_2)$.
- Let T be NIP. Given generically stable measures μ_1 on $\mathbb{M}^{|x_1|}$ and μ_2 on $\mathbb{M}^{|x_2|}$, there is a generically stable product measure of μ_1 and μ_2 (possibly non-unique, can take $\mu_1 \otimes \mu_2$).
- If both μ_1 and μ_2 are smooth, then there is a unique smooth product measure μ.

Main results: Distal Ramsey

Theorem

[Ch., Starchenko] Let T be distal. Then it satisfies:

1. $(*)'$ *For every $\phi(x_1, x_2, y)$ there is some $\varepsilon > 0$ such that: for all $c \in \mathbb{M}^{|y|}$ and all generically stable measures μ_i on $\mathbb{M}^{|x_i|}$ there are some sets $S_i \subseteq \mathbb{M}^{|x_i|}$ definable by an instance of a formula depending just on ϕ, such that $\mu_i(S_i) \geq \varepsilon$ and (S_1, S_2) is $\phi(x_1, x_2, c)$-homogeneous.*
   (Of course, $(*)'$ implies $(*)$ by taking μ_i to be the counting measure concentrated on a finite set A_i.)
2. *Moreover, if T satisfies $(*)'$ just for the counting measures then T is distal.*

- Using it, we generalize the semialgebraic regularity lemma of [J.Fox, M. Gromov, V. Lafforgue, A. Naor, and J. Pach, 2010]:

Main results: Distal regularity lemma

Theorem

[Ch., Starchenko] Let T be distal. For every $\phi(x_1, x_2, y)$ and every $\varepsilon > 0$ there is some $K = K(\varepsilon, \phi)$ such that: for any choice of the parameter $c \in \mathbb{M}^{|y|}$ and any generically stable measures μ_i on $\mathbb{M}^{|x_i|}$, there are $A_0^i, \ldots, A_k^i \subseteq M^{|x_i|}$ uniformly definable depending just on ϕ and ε , and a set $\Sigma \subseteq \{1, \ldots, k\}^2$ such that:

1. $k \leq K$,
2. $\mu\left(\bigcup_{(j,j') \in \Sigma} A_j^1 \times A_{j'}^2\right) \geq 1 - \varepsilon$, *where* μ *is the product measure of* μ_1 *and* μ_2,
3. *for all* $(j, j') \in \Sigma$, *the pair* $\left(A_j^1, A_{j'}^2\right)$ *is* $\phi(x_1, x_2, c)$*-homogeneous.*
4. *Moreover, for a fixed* ϕ *we have* $K(\varepsilon) \leq c_1 \left(\frac{1}{\varepsilon}\right)^{c_2 \log\left(\frac{1}{\varepsilon}\right)}$ *for some* $c_1, c_2 > 0$.

Remarks

- If μ_1, μ_2 also satisfy a certain "uniform non-atomicity" condition, then we can choose the sets in the partition to be of approximately equal size.
- Without requiring definability of the homogeneous subsets $(*)$ holds in ACF_0 and in $\mathrm{ACVF}_{0,0}$: as a model M of $\mathrm{ACVF}_{0,0}$ can be embedded into a model N of RCVF, which is weakly o-minimal, so distal.
- By the same reason, weak $(*)$ holds for all quantifier-free definable graphs in arbitrary (valued) fields of (equi-)characteristic 0.
- There are many further results in the semialgebraic setting relying on $(*)$ and the regularity lemma. For example:

Applications: Erdős-Hajnal property

- Let (G, V) be an undirected graph. A subset $V_0 \subseteq V$ is *homogeneous* if either $(v, v') \in E$ for all $v \neq v' \in V_0$ or $(v, v') \notin E$ for all $v \neq v' \in V_0$.
- A class of finite graphs $\mathcal{G}$ has the *Erdős-Hajnal property* if there is $\delta > 0$ such that every $G \in \mathcal{G}$ has a homogeneous subset of size $\geq |V(G)|^{\delta}$.
- Erdős-Hajnal conjecture: for every finite graph H, the class of all H-free graphs has the Erdős-Hajnal property.
- **Fact.** If $\mathcal{G}$ is a class of finite graphs closed under subgraphs and $\mathcal{G}$ satisfies $(*)$ (without requiring definability of pieces), then $\mathcal{G}$ has the Erdős-Hajnal property.
- Thus, we obtain many new families of graphs satisfying the Erdős-Hajnal conjecture.

Applications: Ramsey numbers

- Let R be a symmetric definable n-ary relation on M^k, and let M be distal.
- A subset $V \subseteq M^k$ is R-mohogeneous if either $(v_1, \ldots, v_n) \in R$ for all pairwise distinct $v_1, \ldots, v_n \in M$ or $(v_1, \ldots, v_n) \notin R$ for all pairwise distinct $v_1, \ldots, v_n \in M$.
- Using the case Erdős-Hajnal property as a basis of induction with $n = 2$, the proof of [D. Conlon, J. Fox, J. Pach, B. Sudakov, A. Suk] for the semialgebraic case gives:

Theorem

There is $c = c(R)$ such that for every m, every finite set of size $m^{m^{\cdot^{\cdot^{\cdot^{m^c}}}}}$ (i.e. $(n-1)$-tower of m's) contains an R-homogeneous subset of size m.

- The bound is tight when k is close to n, but for $k = 1$ it is much smaller [B. Bukh, J. Matoušek].

Some comments on the proof

- The semialgebraic version of $(*)$ is proved using the Clarkson-Shor random sampling technique and a polynomial cutting lemma of Guth and Katz.
- For our argument we replace the polynomial cutting lemma by an abstract version of a cutting obtained using distality and frequency approximation of generically stable measures using the VC-theorem.
- For the converse, we use the average measure of an indiscernible sequence.