Tame geometry and Diophantine approximation

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Tame topology/geometry

Grothendieck '84

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O-minimality is a candidate theory of tame topology:

- Meets many of the criteria suggested by Grothendieck, e.g. allowing stratification and triangulation.
- Contains moduli spaces of algebraic curves and abelian varieties (Peterzil, Starchenko), recently extended to more general variations of Hodge structures (Bakker, Brunebarbe, Klingler, Tsimerman).
- Deep impact on arithmetic geometry (Pila et. al.).

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However, to paraphrase,

"o-minimality" was developed by real geometers and in order to meet the needs of real geometry, not for arithmetic geometry per se.

How tame geometry impacted Diophantine geometry

For $X \subset \mathbb{R}^n$ denote

$$X(g,H) := \{\mathbf{x} \in X : [\mathbb{Q}(\mathbf{x}) : \mathbb{Q}] \leqslant g, H(x) \leqslant H\}. \qquad (1)$$

The story started with the celebrated Pila-Wilkie theorem:

Theorem (Pila-Wilkie)

Let $A \subset \mathbb{R}^n$ be definable in an o-minimal structure. Then for any $\varepsilon > 0$,

$$\#A^{\mathrm{tran}}(g,H) = O_{A,g,\varepsilon}(H^\varepsilon). \qquad (2)$$

- A^{alg} is the union of connected semialgebraic curves in A.
- $A^{\mathrm{tran}} := A \setminus A^{\mathrm{alg}}$.

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It turns out that o-minimally tame sets are, in a sense, tame from the point of view of diophantine geometry as well...

An example: André-Oort for modular curves

Denote

- $\mathbb{H}$ the upper half-space, $\mathcal{F} \subset \mathbb{H}$ the usual $\mathrm{SL}_2(\mathbb{Z})$ fund. domain.
- $Y(1) := \mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H} \cong \mathbb{C}$ the modular curve.
- $j : \mathbb{H} \to Y(1)$ the Klein modular invariant map.

Some points are more special than others

- $p \in Y(1)$ represent an elliptic curve E_p. If E_p has endomorphisms other than $\mathbb{Z}$ then p is called *CM* or *special*.
- $p = j(\tau)$ is special iff τ is a quadratic number.

Consider a curve $V = \{P(p, q) = 0\} \subset Y(1)^2$ over $\mathbb{Q}$.

Some curves are more special than others

V is called *special* if P is a factor of one of the *modular polynomials* Φ_N.

André-Oort for modular curves (cont.)

André-Oort conjecture for $Y(1)^2$

If V is not special then it contains finitely many special pairs (p, q).

This case is due to André.

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This case is due to André. Rough idea of Pila's proof:

- Write $\pi := (j, j) : \mathcal{F}^2 \to Y(1)^2$. Consider $X = \pi^{-1}(V)$.
- X contains no semialgebraic curves (not easy, also uses P-W).
- P-W $\implies$ $\#X(4, H) = O_{X,\varepsilon}(H^\varepsilon)$ for any $\varepsilon > 0$.
- On the other hand, let $(p, q) = j(\tau_1, \tau_2) \in V$ be special and $H = H(\tau_1, \tau_2)$.
- Class field theory $\implies$ (p, q) has H^c Galois conjugates for $c > 0$.
- Each conjugate corresponds to a point in $X(4, H)$. Contradiction for $H \gg 1$.

Limits of o-minimality

This strategy works for many problems, if one can prove suitable lower bounds on Galois orbits. For example:

- David, Masser: for A an abelian variety and $p \in A$ an N-torsion point,

$$[\mathbb{Q}(p) : \mathbb{Q}] \geqslant \text{const}(A) \cdot N^c. \tag{3}$$

- Tsimerman: for $p \in \mathcal{A}_g$ a CM abelian variety (principally polarized),

$$[\mathbb{Q}(p) : \mathbb{Q}] \geqslant \text{const}(g) \cdot \text{disc}(p)^c. \tag{4}$$

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- Transcendence methods and P-W both use auxiliary polynomials to explore algebraic points on transcendental sets.
- We just don't know enough about how general definable sets interact with these auxiliary polynomials.

Pushing the limits

The $O(H^\varepsilon)$ asymptotic in P-W cannot be sharpened in general. But

Conjecture (Wilkie '06)

If Γ is definable in $\mathbb{R}_{\exp}$ then

$$\#\Gamma^{\mathrm{tran}}(g,H) = \mathrm{poly}_\Gamma(g, \log H). \tag{5}$$

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A few years ago Schmidt made the following observation.

If this conjecture holds for a large enough structure, Galois-orbit lower bounds would follow for abelian varieties, Shimura varieties, (...?)

Suppose for example A is an abelian variety and $p \in A$ an N-torsion point of degree g. Let Γ be the graph of the universal cover, $\Gamma^{\mathrm{alg}} = \emptyset$.

- p and its multiples give N points of degree g on Γ.
- Each of the multiples has height $O_A(1)$ downstairs and N upstairs.
- (5) $\implies N \leqslant \#\Gamma^{\mathrm{tran}}(g, N) = \mathrm{poly}_A(g, \log N) \implies N = \mathrm{poly}_A(g)$.

What a general theory might look like

Think of an o-minimal structure X as a collection of definable sets and consider a double increasing filtration

$$X = \cup_{\mathcal{F},D\in\mathbb{N}} X_{\mathcal{F},D}, \qquad X_{\mathcal{F},D} \subset X_{\mathcal{F}+1,D} \cap X_{\mathcal{F},D+1}.$$

Say that sets in $X_{\mathcal{F},D}$ have *format* $\mathcal{F}$ and *degree* D.

Call X sharply o-minimal if for any k sets $A_j \in X_{\mathcal{F}_j,D_j}$:

1. $\cup_j A_j$ has format $\mathcal{F}$ and degree D.
2. $\cap_j A_j$ has format $\mathcal{F}+1$ and degree $\mathrm{poly}_{\mathcal{F}}(k,D)$.
3. A_j^c and $\pi_m(A_j)$ has format $\mathcal{F}_j+1$ and degree $\mathrm{poly}_{\mathcal{F}_j}(D_j)$.
4. The $\#$ of connected components of A_j is $\mathrm{poly}_{\mathcal{F}_j}(D_j)$.
5. Alg. hypersurfaces $\{P=0\}\subset\mathbb{R}^n$ have format n and degree $\deg P$.

where $\mathcal{F} = \max \mathcal{F}_j$ and $D = \sum D_j$.

Some conjectures

$\mathbb{R}_{\text{an}}$ is o-minimal but not sharply: there are analytic $\Gamma \subset [0,1]^2$ that can intersect $\{P = 0\}$ in more than $\text{poly}_\Gamma(\deg P)$ points.

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Conjecture

Arithmetically interesting functions such as universal covers of Shimura varieties live in a sharply o-minimal structure.

We are very far from knowing this! Currently we know two examples:

1. Semialgebraic sets (heavy lifting done by Thom's lemma).
2. B.-Vorobjov: restricted sub-Pfaffian sets (following work of Gabrielov).

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Key issues:

1. Arithmetically interesting functions are often not Pfaffian.
2. Unrestricted exponentials are often required.

Some conjectures (cont.)

Conjecture (Wilkie for sharply o-minimal structures)

If A is definable in a sharply o-minimal structure, with format $\mathcal{F}$ and degree D, then

$$\#A^{\mathrm{tran}}(g,H) = \mathrm{poly}_{\mathcal{F}}(D,g,\log H). \tag{6}$$

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Theorem (B.-Jones-Schmidt-Thomas)

For A as in the conjecture and for every $\varepsilon > 0$,

$$\#A^{\mathrm{tran}}(g,H) = \mathrm{poly}_{\mathcal{F},g,\varepsilon}(D)H^{\varepsilon}. \tag{7}$$

This implies some uniform effective forms of Manin-Mumford.

Beyond Pfaffian functions

Let $\boldsymbol{\xi} = \xi_1, \ldots, \xi_n$ be a collection of commuting vector fields in variety $\mathbb{M}$, all defined over a number field $\mathbb{K}$.

For a variety $V \subset \mathbb{M}$ of codimension n, denote by Σ_V the points where V has improper intersection with the leaf at p.

Beyond Pfaffian functions

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For a variety $V \subset \mathbb{M}$ of codimension n, denote by Σ_V the points where V has improper intersection with the leaf at p.

- Fix a ball $\mathcal{B}$ in one of the leafs, for simplicity assume it is contained in the unit ball in $\mathbb{M}$.
- Denote by δ_V the maximum of the degree deg V and log-height $h(V)$. Similarly for $\delta_{\boldsymbol{\xi}}$.

Ideally we would like to prove sharp o-minimality for structures generated by such sets $\mathcal{B}$.

Nevanlinna-style ideas give a potential approach, but currently with some technical limitations...

Point counting for foliations

Theorem (B.)

Suppose codim $V = n$ *and* $\mathcal{B} \subset \mathbb{B}$. *Then*

$$\#[\mathcal{B} \cap V] \leqslant \operatorname{poly}(\delta_V, \delta_{\boldsymbol{\xi}}, \log \operatorname{dist}^{-1}(\mathcal{B}, \Sigma_V)).$$

- Often applied to flat structures on principal G-bundles.
- In such cases, all leafs of the foliation are equivalent under G-action.
- This often gives good control over Σ_V.

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Theorem (B.)

Suppose $V \cap \mathcal{L}$ *contains no semialgebraic curves for any leaf* $\mathcal{L}$. *Then*

$$\#[\mathcal{B} \cap V](g, H) = \operatorname{poly}(\delta_V, \delta_{\boldsymbol{\xi}}, g, \log H).$$

Full statement more general: images under algebraic maps, blocks...

André-Oort for modular curves revisited

The Schwarzian differential equation

The modular invariant $j : \mathbb{H} \to \mathbb{C}$ satisfies a differential equation

$$S(j) + R(j)(j')^2 = 0, \qquad R(j) := \frac{j^2 - 1968j + 2654208}{2j^2(j-1728)^2}.$$

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- The graph of $j^{\times 2} : \mathbb{H}^2 \to \mathbb{C}^2$ can be realized as a leaf $\mathcal{L}$ of a foliation.
- One can replace counting in $\mathcal{F}^2$ by counting in some big ball $\mathbb{B}$ (either using equidistribution or height bounds).
- Instead of P-W, we can apply our result to $\mathcal{L} \cap \{P(p,q) = 0\}$.

What do we gain? For instance, polynomial dependence on $\deg P$.

Theorem

There is a polynomial time algorithm for computing all special points in V.

We do not know what the algorithm is...

Galois-orbit lower bounds revisited

Let $\pi : \mathfrak{F} \to S$ denote the universal cover of a Shimura variety S. There is:

- a principal G-bundle $P \to S$, a flat structure and a leaf $\mathcal{L} \subset P$,
- a map $\Phi : P \to \check{X} \times S$ such that

$$\Phi(\mathcal{L}) = \text{graph } \pi$$

So we can use foliations to count algebraic point of the graph of π.

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So we can use foliations to count algebraic point of the graph of π.

Theorem (B.-Schmidt-Yafaev (in preparation))

Suppose

$$h(p) = O_\varepsilon(\mathrm{disc}(p)^\varepsilon), \qquad \forall \textit{ special } p \in S. \tag{8}$$

Then $[\mathbb{Q}(p) : \mathbb{Q}] > \mathrm{disc}(p)^c$ *for some* $c > 0$.

- (8) holds for $S = \mathcal{A}_g$ by averaged Colmez (Y-Zh, A-G-H-M).
- We get a new proof of Tsimerman's theorem. Previous proof uses isogeny estimates, doesn't seem to work for general S.
- Corollary: (8) implies André-Oort for general S.

Unlikely intersections in abelian schemes

Consider a family of abelian surfaces $\lambda : A \to \mathbb{C}$ and $C \subset A$ an irreducible curve, both defined over some number field $\mathbb{K}$.

Theorem (Masser-Zannier)

If C is not contained in a proper subgroup of A then contains finitely many torsion points.

- Many extension by Masser, Zannier, Barroero, Capuano, Schmidt...
- Interesting applications around Pell, elementary integration.

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Theorem (B.)

In fact, every torsion $c \in C$ has order at most $\text{poly}(\delta_C, [\mathbb{K} : \mathbb{Q}])$. *In particular, set of torsion points effectively computable in polynomial time.*

- This is the natural expected asymptotic.
- Very likely works for the various extensions...

Proof sketch

Let $A_\lambda := \mathbb{C}^2/\Lambda_\lambda$. Then

1. $c \in C$ is torsion in $A_{\lambda(c)} \iff$ its logarithm in $\mathbb{C}^2$ is a rational combination of the generators of $\Lambda_{\lambda(c)}$.
2. The order of torsion N is roughly the height of the coefficients.
3. P-W: the number of $c \in C$ that are N-torsion is $O_C(N^\varepsilon)$.
4. If $c \in C$ is N-torsion then $[\mathbb{Q}(c) : \mathbb{Q}] \geqslant \mathrm{const}(C) N^d$ for $d > 0$.
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Foliations everywhere!

1. The generators of Λ_λ are given by complete hyperelliptic integrals, satisfy homogeneous Picard-Fuchs type equations.
2. The $A_{\lambda(c)}$-logarithm of c is an incomplete hyperelliptic integral, satisfies inhomogeneous Picard-Fuchs type equation.

Foliation point-counting replaces $O_C(N^\varepsilon)$ by $\mathrm{poly}(\delta_C, [\mathbb{K} : \mathbb{Q}], \log N)$.

Pell's equation over $\mathbb{C}[t]$

Consider Pell's equation for fixed $D \in \mathbb{C}[X]$

$$A^2 - DB^2 = 1$$

and try to solve with non-zero $A, B \in \mathbb{C}[X]$. In the integers this is always solvable unless D is a perfect square. How about with polynomials?

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Theorem (Masser-Zannier '15)

Let $D = X^6 + X + t$. Pell's equation is only solvable for finitely many t.

Actually there is a criterion that works for any $D \in \mathbb{C}[X, t]$, we consider this example for simplicity.

Theorem (B.)

The set of t is effectively computable in polynomial time (with D as input).

Reduction to torsion points

Fox fixed $t \in \mathbb{C}$ consider the hyperelliptic curve $C_t := \{Y^2 = D(t)\}$. If a polynomial solution $A, B \in \mathbb{C}[x]$ to Pell exists,

$$(A - YB)(A + YB) = 1, \tag{9}$$

then $A - YB$ is a regular function on C_t without poles or zeros, except at the two points at infinity.

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then $A - YB$ is a regular function on C_t without poles or zeros, except at the two points at infinity.
Thus in the Jacobian A_t of C_t,

$$0 = [A - YB] = m(\infty_1 - \infty_2). \tag{10}$$

In particular, $s(t) := \infty_1 - \infty_2 \in A_t$ is torsion.

Conclusion

By effective M-Z we can find all t where this happens.

Back to André-Oort: effectivity

There is only one case where André-Oort is known effectively.

Theorem (Kühne '12, Bilu-Masser-Zannier '13)

If $V \subset \mathbb{C}^n$ is a curve defined over $\mathbb{Q}^{\mathrm{alg}}$, one can effectively find all special points in V.

Both proofs are similar, exploit two crucial facts:

- All CM-points with discriminant D are Galois conjugate, and $\tau = \frac{(0 \text{ or } 1)+i\sqrt{D}}{2}$ gives the one closest to the cusp.
- There are strong transcendence estimates showing that elliptic logarithms cannot be well approximated by algebraic numbers.

Much less is known about the latter for general Shimura varieties, and the former fails completely $\implies$ basically nothing known for other Shimuras.

In $\mathbb{C}^n$ some partial results known with $\dim V > 1$.

A new effective result

Theorem (B.-Masser, in preparation)

Let S be a Hilbert modular variety and $V \subset S$ be a non-compact curve defined over $\mathbb{Q}^{\mathrm{alg}}$. Then one can effectively find all special points in V.

Non-compactness was automatic for $V \subset \mathbb{C}^n$, but not for HMVs...

Proof strategy:

1. G-functions bound heights of special points in terms of their degrees.
2. Masser-Wüstholz endomorphism estimates bound the discriminant of special points in terms of the degree (Galois lower bound).
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Here we must do unrestricted counting on the entire fundamental domain.

We develop an approach, currently only for curves, based on theory of Q-functions (B.-Novikov-Yakovenko). Complex cells start to appear...