Wilkie's conjecture for restricted elementary functions

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Some notations and a question

For a reduced fraction p/q we define it's *height* $H(p/q) := \max(|p|, |q|)$.
For a vector $\mathbf{x} \in \mathbb{Q}^n$ we define $H(\mathbf{x})$ to be the maximal height of its coordinates.
Let $A \subset \mathbb{R}^n$. We define $A(\mathbb{Q})$ to be $A \cap \mathbb{Q}^n$ and

$$A(\mathbb{Q}, H) := \{\mathbf{x} \in A : H(\mathbf{x}) \leqslant H\}.$$

Question

For suitable sets A, can we understand the asymptotic growth of $\#A(\mathbb{Q}, H)$ in terms of H?

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Bombieri-Pila on analytic curves

Let $\mathbf{f} := (f_1, f_2) : I \to \mathbb{R}^2$ be an analytic map and write $\Gamma = \mathbf{f}(I)$.

Theorem (Bombieri-Pila '89 (+Pila '91))

Suppose Γ does not belong to an algebraic curve in $\mathbb{R}^2$. Then for every $\varepsilon > 0$ there exists $C = C(\Gamma, \varepsilon)$ such that

$$\#\Gamma(\mathbb{Q}, H) \leqslant C \cdot H^{\varepsilon}.$$

- The same conclusion holds for any compact analytic irreducible curve $\Gamma \subset \mathbb{R}^2$ since it can be parametrized by images $\mathbf{f}(I)$ as above.
- The conclusion can certainly fail if Γ is algebraic!
- Asymptotic is essentially optimal for general analytic curves.

General outline: show that all points of $\Gamma(\mathbb{Q}, H)$ belong to $\sim H^{\varepsilon}$ algebraic curves of degree $d = d(\varepsilon)$.

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Interpolation determinants

Question

How do we show a collection of points $P = \{\mathbf{p}\} \subset \mathbb{R}^2$ all satisfy an algebraic relation $P(x, y) = 0$ of degree d?

Linear algebra: let $\mu = (d+1)(d+2)/2$ be the dimension of the space of polynomials P above. Vanishing of P at a given point $\mathbf{p}$ is a linear condition on this space. When do they have a common kernel? If and only if for every $\mathbf{p}_1, \ldots, \mathbf{p}_\mu \in P$ the following determinant vanishes,

$$\Delta^d(\mathbf{p}_1, \ldots, \mathbf{p}_\mu) := \det(\mathbf{x}^\alpha(\mathbf{p}_j))_{j=1,\ldots,\mu,|\alpha|\leqslant d}. \tag{1}$$

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Bombieri-Pila: the main estimate

We illustrate the main estimate in the holomorphic setting.

Proposition

Let $z_1, \dots, z_\mu \in D$ with $|z_j| < \delta$ and let $g_1, \dots, g_\mu : D \to \mathbb{C}$ be holomorphic and bounded by 1. Then $|\det(g_i(z_j))| \lesssim \delta^{\sim\mu^2}$.

Proof.

Expand each $g_i = \sum c_{ik} z^k$ as a Taylor series and then expand

$$\det \begin{pmatrix} g_1(z_1) & \cdot & g_\mu(z_1) \\ & \ddots & \\ g_1(z_\mu) & \cdots & g_\mu(z_\mu) \end{pmatrix} \tag{2}$$

by multilinearity in the columns. Main point: a summand where the same monomial z^k is repeated in two columns vanishes identically! So each o the monomials $z^0, z^1, \dots$ can appear at most once. Since the z_j points have modulus at most δ each summand has order at least $\sim \delta^{1+\cdots+\mu}$. □

Idea of Bombieri-Pila

Suppose J is an interval of length δ, and let P be the set of points of height H in $\mathbf{f}(J)$.

- As above with $x = f_1, y = f_2$ we get $|\Delta^d| \lesssim \delta^{\sim\mu^2} = \delta^{\sim d^4}$.
- Either $\Delta^d = 0$ or $|\Delta^d| \gtrsim H^{\sim(-d^3)}$, since we know the common denominator of the determinant Δ^d.

So, if $\delta \lesssim H^{-1/d}$ we must have $\Delta^d = 0 \implies P$ is contained in an algebraic hypersurface of degree d.
Now for given ε choose $d > \varepsilon^{-1}$ and subdivide I into $\sim H^\varepsilon$ intervals of length $\delta = H^{-\varepsilon}$, and apply the above to each.

Conclusion

$\Gamma(\mathbb{Q}, H)$ is contained in H^ε algebraic curves $\{P_j = 0\}$, each of degree $d = d(\varepsilon)$.

Final claim: the number of intersections $\Gamma \cap \{P_j = 0\}$ is bounded by some constant depending only on d! Follows from general finiteness properties of the analytic category, e.g. Gabrielov's theorem.

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Going beyond curves: new problems

Let $A \subset \mathbb{R}^n$ be a set of higher dimension and suppose it is transcendental. Can we expect $\#A(\mathbb{Q}, H) = O(H^\varepsilon)$? No! A might still contain algebraic curves, which contain many rational points.

Let the *algebraic part* A^{alg} be the union of all positive-dimensional connected semialgebraic sets contained in A: this is the part we have no hope to control. Let the *transcendental part* be $A^{\mathrm{trans}} := A \setminus A^{\mathrm{alg}}$. Hope: extend everything to A^{trans}.

Warning: A^{alg} can be extremely complicated, for instance if

$$A = \{(t/s, e^s, e^t) : s, t \in \mathbb{R}, s \neq 0\} \subset \mathbb{R}^3$$

then A is a transcendental surface, but for each $r \in \mathbb{Q}$ it contains the algebraic curve (r, t, t^r).

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Pila-Wilkie Theorem

For the general statement, we need a tame geometric category to avoid pathologies in the set A. The general framework is provided by the notion of an *o-minimal* structure. For example, this contains all subsets of the unit cube defined by analytic equalities and inequalities (and much much more!).

Theorem (Pila-Wilkie, '06)

Let A be definable in an o-minimal structure. Then for every $\varepsilon > 0$ there exists $C = C(A, \varepsilon)$ such that

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Idea of proof for P-W Theorem

By induction on $m := \dim A$:

1. Cover A by finitely many images of maps $\mathbf{f} : I^m \to A$.
2. For each map, construct H^ε hypersurfaces S_α of degree $d = d(\varepsilon)$ containing the rational points of height H in the image.
3. Continue inductively for each intersection $A \cap S_\alpha$.

Step 2 is a relatively straightforward generalization of the Bombieri-Pila method. To carry out steps 1 and 3 we must show not only that A can be parameterized, but also that the intersections $A \cap S_\alpha$ can be parameterized with the number of maps $\mathbf{f}$ (with bounded norms) independent of S_α. Unfortunately this is impossible to accomplish using analytic maps! With C^r-maps it is possible: for semialgebraic families this is the Yomdin-Gromov theorem, generalized to o-minimal structures by P-W.

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Where to go next?

The asymptotic $\#A^{\mathrm{trans}}(\mathbb{Q}, H) = O(H^\varepsilon)$ is optimal even for analytic curves, what else can we hope for?

- Effectivity: if we could compute the constant in $C \cdot H^\varepsilon$ then we could potentially effectivize many of the diophantine applications of the P-W theorem.
- Better bounds for "natural" geometric structures: the examples where you really have $\sim H^\varepsilon$ points of height H involve specifically crafted analytic functions. Maybe something better holds if we only allow natural functions like e^z.

Both directions are interesting and studied by many authors: Boxall, Cluckers, Comte, Habegger, Jones, Masser, Miller, Pila, Thomas, Wilkie, Yomdin, . . .

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Wilkie's conjecture

A prominent conjecture on improvements to the asymptotic is due to Wilkie:

Conjecture

Suppose A is definable in $\mathbb{R}_{\exp}$. Then there exists $C = C(A)$ and $\kappa = \kappa(A)$ such that

$$\#A(\mathbb{Q}, H) \leqslant C \cdot (\log H)^\kappa.$$

Definable in $\mathbb{R}_{\exp}$ means that A can be expressed using $+, \cdot, <$, real constants, the real exponential e^x, and logical connectives and quantifiers. For example:

$$A = \{(x, y, \alpha) : x = y^\alpha\} = \{(x, y, \alpha) : \exists z : y = e^z \wedge x = e^{\alpha \cdot z}\}$$

Only some special cases of curves and surfaces are known.

Restricted elementary functions

Consider the structure $\mathbb{R}^{\mathrm{RE}}$: similar to $\mathbb{R}_{\exp}$, but instead of allowing e^x we allow *restricted* elementary functions: $\exp|_{[0,1]}$ and $\sin|_{[0,\pi]}$.

- Not important which intervals we restrict to because of the sum rules for $\exp, \sin$.
- Thus $\mathbb{R}^{\mathrm{RE}}$ defines the graph of the complex exponential e^z on any compact set.

Theorem (B., Novikov)

Let A be definable in $\mathbb{R}^{\mathrm{RE}}$. Then there exists $C = C(A)$ and $\kappa = \kappa(A)$ such that

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The constant κ is computed and C is computable in principle.

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What's so special about restricted elementary functions?

The restricted elementary functions, as well as their complexification, are *Pfaffian*: they can expressed as polynomial combinations of solutions of a triangular system of algebraic differential equations:

$$\begin{aligned}(e^x)' &= e^x \\ (\tan x)' &= 1 + \tan^2 x \\ (\cos x)' &= -(\tan x) \cdot (\cos x) \\ (\sin x)' &= \cos x\end{aligned}$$

The geometric complexity of sets defined by Pfaffian functions can be explicitly bounded from above in terms of the degrees of the equations, with bounds depending polynomially on the degrees. This is Khovanskii's theory of *Fewnomials*, extended to semi-Pfaffian and sub-Pfaffian sets by Gabrielov-Vorobjov.

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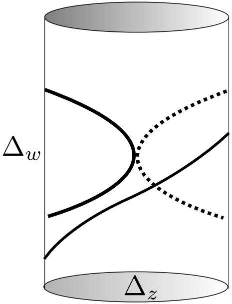
Weierstrass Polydiscs

The main difficulty in carrying out the P-W proof in a more sharp manner is the construction of C^r-smooth parametrizations. To overcome this we introduce the following complex-analytic notion:

Definition (Weierstrass Polydisc)

If $X \subset \mathbb{C}^n$ is a purely m-dimensional analytic set, we say that a polydisc $\Delta = \Delta_z \times \Delta_w$ is a Weierstrass polydisc for X if $\dim \Delta_z = m$ and $X \cap (\Delta_z \times \partial\Delta_w) = \emptyset$.



When restricting to a Weierstrass polydisc, the projection from X to Δ_z is a finite ramified map. We denote its degree by $e(X, \Delta)$.

Interpolation on Weierstrass polydiscs

Let $f : \Delta^{1/3} \to \mathbb{C}$ be holomorphic on a polydisc three times larger than Δ.

Proposition

There exists a function P holomorphic in Δ_z and polynomial of degree at most $e(X, \Delta) - 1$ in each of the w variables which agrees with f on $X \cap \Delta$. Moreover $\|P\|_\Delta \leqslant 3^{n-m} \|f\|_{\Delta^{1/3}}$.

Proof for $\dim \mathbf{w} = 1, \dim X = \dim \mathbf{z} = 0$

With $h(w)$ the monic polynomial vanishing at the points of $X \cap \Delta$,

$$P(w) = \frac{1}{2\pi i} \oint_{\partial \Delta_w} \frac{f(\eta)}{h(\eta)} \cdot \frac{h(\eta) - h(w)}{\eta - w} \mathrm{d}\eta. \tag{4}$$

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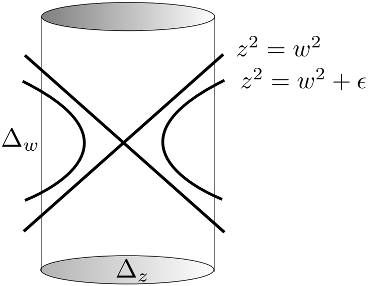
Example

Suppose $X \subset \mathbb{C}^2$ is given by $z^2 = w^2 + \varepsilon$ with $|\varepsilon| < 1$. Then $D_1 \times D_2$ is a Weierstrass polydisc. Given $f(z, w)$, we can construct P by replacing each w^2 in the Taylor expansion by $z^2 - \varepsilon$.



Exploring rational points in Weierstrass polydiscs

Proposition (following Bombieri-Pila)

One can construct an algebraic hypersurface of degree $(e(X,\Delta)+\log H)^{O(1)}$ *containing* $(X\cap\Delta^2)(\mathbb{Q},H)$.

The proof is analogous to the original Bombieri-Pila proof. We consider the interpolation determinants of the points of height H in $X\cap\Delta^2$, and get competing lower and upper bounds.

Main point

We can use the previous polynomial interpolation result instead of Taylor series expansions in deriving the main estimate! This is where C^r-smoothness, allowing Taylor approximation of order r, is replaced by a holomorphic argument.

Bombieri-Pila: the main estimate for $X := z^2 - w^2 = \varepsilon$

Proposition

Let $z_1, \dots, z_\mu \in X \cap \Delta^{1/\delta}$ *and let* $g_1, \dots, g_\mu : \Delta \to \mathbb{C}$ *be holomorphic and bounded by* 1*. Then* $|\det(g_i(z_j))| \lesssim \delta^{\sim \mu^2}$*.*

Proof.

Expand each $g_i = (\sum b_{ik} z^k) + (\sum c_{ik} z^k) w$ and then expand

$$\det \begin{pmatrix} g_1(z_1) & \cdot & g_\mu(z_1) \\ & \ddots & \\ g_1(z_\mu) & \cdots & g_\mu(z_\mu) \end{pmatrix} \tag{4}$$

by multilinearity in the columns. Main point: a summand where the same monomial z^k or $z^k w$ is repeated in two columns vanishes identically! So each monomial $z^0, w, z, zw, z^2, \dots$ can appear at most once. Since the monomial $z^k w$ has order at least δ^k in $\Delta^{1/\delta}$ each summand has order at least $\delta^{1+\cdots+\mu/2}$. □

Proof structure

Key ideas for the proof of the restricted Wilkie conjecture:

1. Start with some set A defined using restricted elementary formulas.
2. Build a suitable complexification X of A using a quantifier elimination result of van den Dries. The equations for X will be Pfaffian of some degree β.
3. Instead of parametrizations $\mathbf{f} : I^m \to X$, construct a covering of X by $\text{poly}(\beta)$ Weierstrass polydiscs Δ_j.
4. For each Δ_j construct an algebraic hypersurface S_j of degree $\text{poly}(\beta + \log H)$ containing $(X \cap \Delta_j)(\mathbb{Q}, H)$.
5. Continue by induction on each $X \cap S_j$ – everything remains polynomial in β and $\log H$.

Steps 1-2 work for arbitrary subanalytic sets and give a proof of the P-W theorem for $\mathbb{R}_{\text{an}}$. Step 3 is the heart of the argument for polylog bounds.

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Covering by Weierstrass polydiscs

We consider the following elementary question which contains all the key ideas for the general statement.

Question

Let $\Gamma \subset \mathbb{C}^2$ be an algebraic curve of degree d. Find a Weierstrass polydisc centered at the origin, contained in the unit ball and containing a ball of radius $1/\operatorname{poly}(d)$ around the origin.

First main idea:

- Suppose we find a ball B of radius $1/\operatorname{poly}(d)$ inside the unit ball which is disjoint from $S^1 \cdot \Gamma$.
- WLOG the center is at some point $(0, w_1)$ of modulus $r = 1/\operatorname{poly}(d)$ on the w-axis, and B contains every point of the form (z, w_1) where $|z| \leqslant r$.
- Then Γ is automatically disjoint from $S^1 \cdot B$ which contains $\{|w| = r\} \times \{|z| \leqslant r\}$, i.e. we get a Weierstrass polydisc as needed.

Covering by Weierstrass polydiscs (cont.)

Question

How can we find a large ball disjoint from $S^1 \cdot \Gamma$?

The answer is given by a simple geometric idea related to metric entropy, called Vitushkin's formula. Once again we consider a simpler question to make drawing easier:

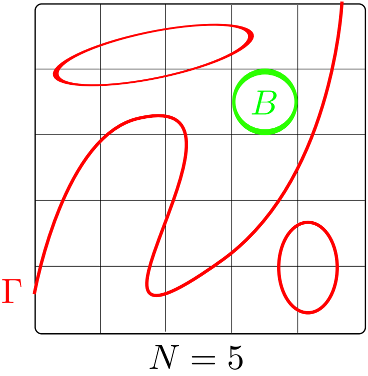
Question

Let $\Gamma \subset \mathbb{R}^2$ be a real algebraic curve of degree d. Find a ball of radius $1/\operatorname{poly}(d)$ in the unit square which is disjoint from Γ.

In our case, $S^1 \cdot \Gamma$ is a real-codimension-one set in $\mathbb{R}^4$ and essentially the same proof works. The proof is a picture!

Covering by Weierstrass polydiscs (end)

1. Divide $[0,1]^2$ into an $N \times N$ grid.
2. The number V_0 of conn. components of Γ is at most d^2 (Harnack).
3. The number V_1 of intersections between Γ and a line in the grid is at most d (Bezout).
4. Γ meets at most $V_0 + 2NV_1$ cells out of the N^2 cells in the grid.
5. Conclusion: once $N \gg d$ we can find a cell that doesn't meet Γ.



In dimension n with Γ any algebraic (or even sub-Pfaffian) set we will have N^n cells, and Γ meets at most $N^{n-1} \cdot \text{poly}(d)$ of them.

Future directions

Beyond restricted elementary functions, a similar strategy would probably work for elliptic and abelian functions (using work of Macintyre and of Bianconi). These (in addition to $\mathbb{R}^{\mathrm{RE}}$) are the functions important for applications around Manin-Mumford.

It is reasonable to hope that the same should also hold for *modular* functions, e.g. Klein's modular invariant $j(z)$, and automorphic functions of other Fuchsian groups. These are the functions important for applications around the André-Oort conjecture. The first two steps of the proof probably extend, but modular functions are not Pfaffian (to our knowledge)!

Modular function do satisfy certain non-Pfaffian systems of algebraic equations. Analyzing these systems might lead to effective estimates for the P-W constant and, more optimistically, to polylogarithmic bounds as in the Wilkie conjecture...

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Future directions

Beyond restricted elementary functions, a similar strategy would probably work for elliptic and abelian functions (using work of Macintyre and of Bianconi). These (in addition to $\mathbb{R}^{\mathrm{RE}}$) are the functions important for applications around Manin-Mumford.

It is reasonable to hope that the same should also hold for *modular* functions, e.g. Klein's modular invariant $j(z)$, and automorphic functions of other Fuchsian groups. These are the functions important for applications around the André-Oort conjecture. The first two steps of the proof probably extend, but modular functions are not Pfaffian (to our knowledge)!

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