Asymptotic Analysis of Skolem's exponential functions

Alessandro Berarducci
Joint work with Marcello Mamino

PARIS 15 January 2021

Skolem functions

$$1, x \in Sk$$

$$f, g \in Sk \implies f+g, \quad fg, \quad f^g \in Sk$$

Order

$$f < g \iff \exists n \forall x > n \quad f(x) < g(x)$$

$$((x+2)^{2^x} + (3^{x^2} + x2^x))^{(x+1)^x}$$

$$(x+1)^x < x^{x+1}$$

Is the order decidable?

Normal form

$$f = \sum_i x^i \prod_j f_{ij}^{g_{ij}}$$

REDUCTIONS

$$1^f \to 1$$

$$f^1 \to f$$

$$f^{g+h} \to f^g f^h$$

$$f^{(gh)} = (f^g)^h$$

$$(fg)^h \to f^h g^h$$

Assume $f_{ij}^{g_{ij}}$ is irreducible.

Is the normal form unique?

Or can it happen that $f^g = f_1^{g_1} + f_2^{g_2}$?

Richardson (1969)

Given f, g we can compute $n \in \mathbb{N}$ such that $f - g$ has at most n zeros.

So the equality of Skolem functions is decidable.

Ber-Servi 2003: effective o-minimality of $\mathbb{R}_{\exp}$

Ehrenfeucht (1973): there are no infinite descending chains $f_0 > f_1 > f_2 > \ldots$ of Skolem functions.

$x^{x+1} > (x+1)^x > x^x > 100^x > 2^x > x^{100} > x^2 > 100x > \ldots$

after finitely many steps we must reach 1.

Skolem (1956): there is an order isomorphism from the ordinal

$$\varepsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \omega^{\omega^{\omega^\omega}}, \ldots\}$$

to a subset of the Skolem functions:

$$\alpha = \omega^{\alpha_1} + \ldots + \omega^{\alpha_n} \;\mapsto\; f_\alpha = x^{f_{\alpha_1}} + \ldots + x^{f_{\alpha_n}}$$

Thus the order type of $(Sk, <)$ is at least ε_0.

Is it equal?

$\mathbb{N}[x]$ is a subset of the Skolem functions of order type ω^ω:

$$a_0 + a_1 x + \ldots + a_n x^n \;\mapsto\; \omega^n a_n + \ldots + \omega a_1 + a_0$$

$$(x+1)^x > x^x \text{ but } (\omega+1)^\omega = \omega^\omega$$

$$f \in Sk \implies f < 2^{\vdots^{2^{2^x}}} \text{ for some tower of powers}$$

van den Dries, Levitz (1984)

The set of Skolem functions $< 2^{2^x}$ has order type ω^{ω^ω}.

Note that if $f^g < 2^{2^x}$ then $g < x^n$ (unless $f = 1$).

So the fragment $< 2^{2^x}$ coincides with the fragment generated by $1, x, f+g, fg, f^x$ (only allow exponent x).

Exponential constants

$\mathbb{E}^+$ is the smallest set of reals containing 1 and closed under $+, \cdot, ^{-1}$, exp.

$\mathbb{E} = \mathbb{E}^+ - \mathbb{E}^+$ = exponential constants.

The coefficients of the surreal expansion [sorry, this is defined later!] of a Skolem function belong to $\mathbb{E}$ and the first coefficient belongs to $\mathbb{E}^+$.

R. Gurevic (1986) proved that the problem of deciding $<$ for Skolem functions $< 2^{2^x}$ is Turing equivalent to the problem of deciding the equality of two exponential constants.

Laurent series in $x \to +\infty$

$$f(x) = a_3x^3 + a_2x^2 + a_1x \ + \ a_0 \ + \ \frac{a_{-1}}{x} + \frac{a_{-2}}{x^2} + \dots$$

There is a “purely infinite part”, a constant part, an infinitesimal part.

MAIN LEMMA of van den Dries - Levitz 1984

Fix a Skolem function $g < 2^{2^x}$.

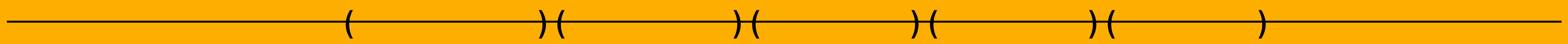
Given $n \in \mathbb{N}$ and $a_0, a_1, \ldots, a_{n-1}$ in $\mathbb{R}$ there is a discrete set $D = \{r_i \mid i \in \mathbb{N}\} \subset \mathbb{R}$ with $r_i \to +\infty$ such that if f is a Skolem functions such that

$$\frac{f}{g} \sim a_0 + a_1 x^{-1} + \ldots + a_{n-1} x^{n-1} + \sum_{i=n}^{\infty} a_i x^{-i} \qquad \text{(Laurent expansion)}$$

then $a_n \in D$.

In particular, taking $n = 0$, the set of possible limits $\lim_{x \to \infty} \frac{f}{g}$ is a discrete subset of $\mathbb{R}$ of order type ω.

Consider partitions of the ordered set of Skolem functions into convex intervals



If $\lim f/h = 1$ we say that f, h are asymptotic.

The main lemma shows that there are ω asymptotic classes within each archimedean class.

From this (plus a complex induction) they deduce the bound ω^{ω^ω} on the fragment below 2^{2^x}.

The proof of the main lemma is by induction on the formation of Skolem expressions. At some point one needs to control how the Laurent expansion is affected raising the expressions to the power x. This involves the following computation.

If $\frac{f}{g} \sim 1 + a_1 x^{-1} + a_2 x^{-2} + a_3 x^{-3} + \ldots$

Then $\frac{f^x}{g^x} \sim e^{a_1}(1 + q_1(a_1, a_2)x^{-1} + q_2(a_1, a_2, a_3)x^{-2} + \ldots)$

where q_i are fixed polynomials over $\mathbb{Q}$.

The above computation shows how to control the Laurent expansion when $\frac{f}{g}$ is raised to the power x.

For bigger exponents Laurent expansions do not give enough information:

Suppose $\frac{f}{g} = 1 + \frac{1}{2^x}$ (looking at the Laurent part you only see "1")

Then $\frac{f^{2^x}}{g^{2^x}} = e - \frac{e}{2^{x+1}} + \ldots$ (looking at the Laurent part you see "e")

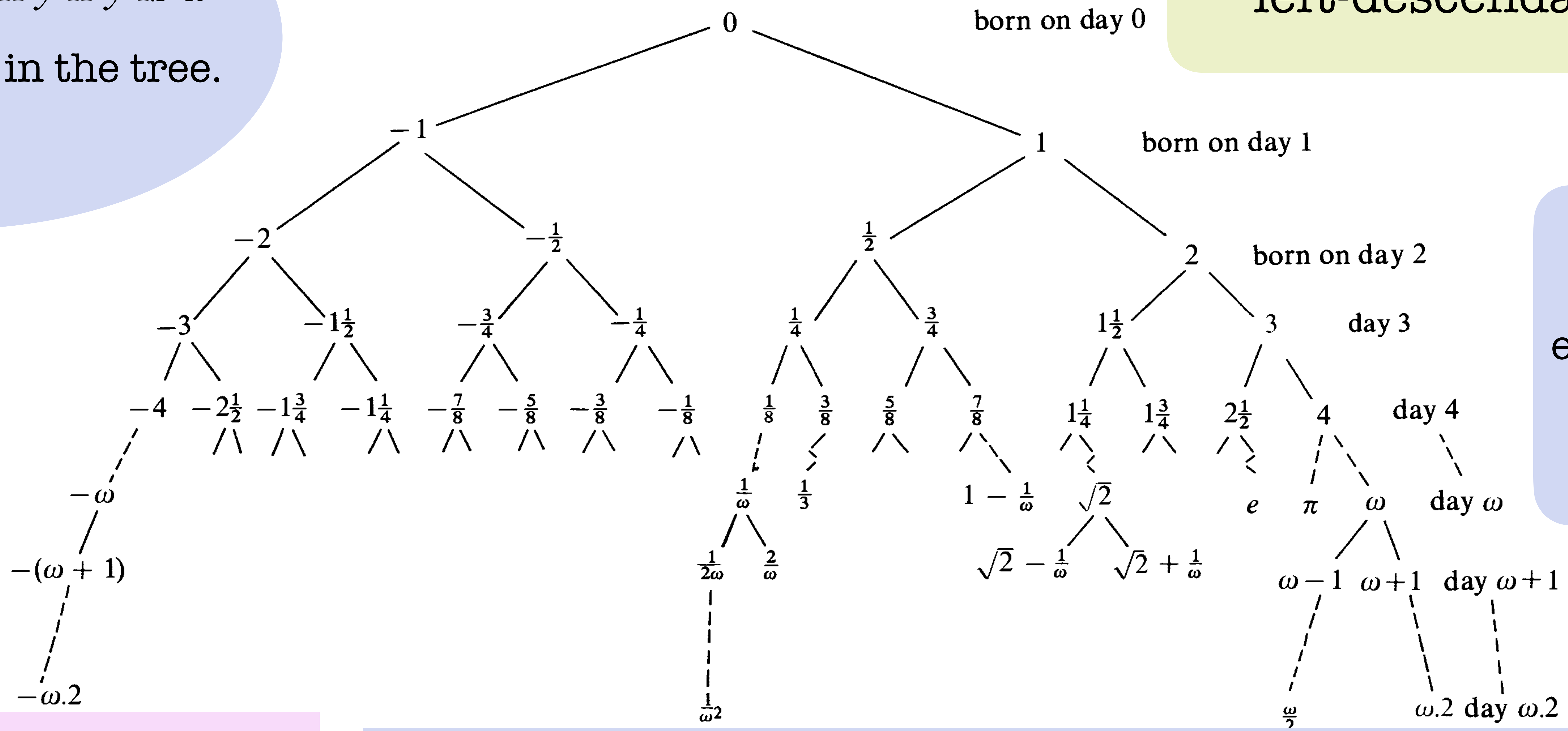
This is why the analysis of van den Dries - Levitz does not extend beyond the fragment $< 2^{2^x}$. We can overcome the obstacle by considering "surreal expansions" which take into account exponentially small terms coming "after" the Laurent expansion.

The surreal numbers are the nodes of a transfinite binary tree

x is simpler than y if y is a descendant of x in the tree.

$x <$ right descendants of x;

left-descendants of x are $< x$



x^L, x^R are generic elements simpler than x with $x^L < x < x^R$

Given two sets of surreals $A < B$

$A \mid B$ is the simplest surreal x with

$A < x < B$

$$x + y = \{x^L + y,\ x + y^L\} \mid \{x^R + y,\ x + y^R\}$$

$$xy = \{x^L y + x y^L - x^L y^L,\ x^R y + x y^R - x^R y^R\} \mid \{x^L y + x y^R - x^L y^R,\ x y^L + x^R y - x^R y^L\}$$

The surreals form a real closed field containing $\mathbb{R}$

A surreal monomial is the simplest positive surreal in its archimedean class

$1, \quad 1/\omega, \quad \omega$ are monomials, and so is ω^n for $n \in \mathbb{Z}$

Every non-zero surreal x can be written as $x = r\mathfrak{m} + o(\mathfrak{m})$ where $r \in \mathbb{R}$ and $\mathfrak{m}$ is a monomial.

Iterating, we can write every surreal as a sum $\sum_{i<\alpha} r_i \mathfrak{m}_i$ where

α is an ordinal.

$r_i \in \mathbb{R}$,

$(\mathfrak{m}_i)_{i<\alpha}$ is a decreasing sequence of surreal monomials.

For α limit, $x = \sum_{i<\alpha} r_i \mathfrak{m}_i$ is the simplest surreal such that $x - \sum_{i<\beta} r_i \mathfrak{m}_i$ is in the archimedean class of $\mathfrak{m}_\beta$ for all $\beta < \alpha$.

A surreal is purely infinite if all monomials in its support are > 1

$\mathbb{S}$ = Surreal numbers of countable birthday.

$\mathbb{S}$ is an ordered field containing $\mathbb{R}$ and an element $\omega > \mathbb{R}$

$\mathbb{S} = \Gamma + \mathbb{R} + o(1)$ (purely infinite + constant + infinitesimal)

There is $\exp : \mathbb{S} \to \mathbb{S}^{\geq 0}, x \mapsto e^x$, with inverse $\log$, making $\mathbb{S} \models T_{\exp}$

$\exp(\Gamma)$ = surreal monomials.

$\omega, \ \log(\omega), \ \log(\log(\omega)), \ \dots$ are monomials.

$(\gamma_i)_{i<\alpha}$ decreasing sequence in $\Gamma \Longrightarrow \sum_{i<\alpha} r_i e^{\gamma_i} \in \mathbb{S}$ (where $r_i \in \mathbb{R}, \alpha < \omega_1$)

If moreover $\gamma_i > 0$ for all i, $\Longrightarrow \sum_{i<\alpha} r_i e^{\gamma_i} \in \Gamma$

$$\exp(1/\omega) = \sum_n \frac{\omega^{-n}}{n!}$$

$$\exp(\omega) \neq \sum_n \frac{\omega^n}{n!}$$

$$\log(1+\omega^{-1}) = \sum_{n=1}^{\infty} \omega^{-n}/n$$

$$f^g = e^{g \log f}$$

A Skolem function $f(x)$ is **determined** by its surreal value at $x = \omega$.

Thus we get an embedding $f(x) \mapsto f(\omega) = \sum_i r_i e^{\gamma_i} \in \mathbb{S}$

This can help to decide $f(x) < g(x)$.

For instance to show that $(x+1)^x < x^{x+1}$ we can write

$$(\omega+1)^\omega = e\omega^\omega + \frac{e}{2}\omega^{\omega-1} + \dots$$

$$\omega^{\omega+1} = \omega\omega^\omega$$

This does not suffice to decide the order on Sk.

- need to be able to compare the real coefficients of the expansions.
- it may happen that the inequality between f and g can only be detected after transfinitely many terms of the expansion.

Nevertheless, we can use the map $f(x) \mapsto f(\omega)$ to extend the Main Lemma of van den Dries - Levitz.

Main result (extension of the main lemma of van den Dries and Levitz to the whole class of Skolem functions).

Fix a Skolem function $g(x)$.
Given $\alpha < \omega_1$, $(a_i)_{i<\alpha}$ in $\mathbb{R}$ and $(\gamma_i)_{i\leq\alpha} \in \Gamma$, there is a discrete set $D = \{r_i \mid i \in \mathbb{N}\} \subset \mathbb{R}$ with $r_i \to +\infty$ such that if $f(x)$ is a Skolem function and

$$\frac{f(\omega)}{g(\omega)} = \sum_{i\leq\alpha} a_i e^{\gamma_i} + o(e^{\gamma_\alpha})$$

then $a_\alpha \in D$.
Since $\lim_x \dfrac{f(x)}{g(x)} = r \in \mathbb{R}$ if and only if $\dfrac{f(\omega)}{g(\omega)} = r + o(1)$ it follows that the set of possible limits $\lim_{x\to\infty} \dfrac{f}{g}$, as f varies, is a discrete subset of $\mathbb{R}$ of order type ω.

For $f, g \in Sk,\ 1 \leq c \in \mathbb{S}$, we have:

$$\lim \frac{f^c}{g^c} \in \mathbb{R}^* \iff c(f-g) = O(g). \text{ Write } f \asymp_c g.$$

$$\lim \frac{f^c}{g^c} = 1 \iff c(f-g) = o(g). \text{ Write } f \sim_c g.$$

The main theorem says that there are ω classes mod $\sim_c$ within any class mod $\asymp_c$.

Despite the fact that the main result holds for the whole class of Skolem functions, we have not been able to use it to solve Skolem's conjecture. We can however get the following bounds.

- the order type of the set of Skolem function $< 2^{3^x}$ is bounded by $\omega^{\omega^{\omega^{\omega}}}$.
- the order type of the set of Skolem functions $< 2^{(n+1)^x}$ is bounded by a tower $\omega^{\vdots^{\omega^{\omega^{\omega}}}}$ ($n+2$ occurrences of ω).
- the order type of the set of Skolem functions $< 2^{x^x}$ is bounded by ε_0.

There is ample room for improvement: to get the above bounds we have used a very rough estimate for the number of equivalence classes mod $\asymp_{2^x}$ within the fragment $< 2^{2^x}$.

Thanks