Topology of definable groups

in tame structures

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Géométrie et Théorie des Modèles

Paris, École Normale Supérieure
6 Mai 2011

O-minimal structures

Definition

A structure $M = (M, <, \dots)$ is **o-minimal** if every definable subset of M is a finite union of points and intervals (a, b) with $a, b \in M \cup \{\pm\infty\}$.

Example

1. $(\mathbb{R}, <, +, \cdot)$.
2. Any real closed field $(R, <, +, \cdot)$.
3. $(\mathbb{R}, <, +, \cdot, \exp, \sin_{|[0,1]})$
4. $\mathbb{R}_{an}$

Triangulations

Fix an o-minimal structure M. We always assume that M expands an ordered field. We recall the following basic result.

Theorem ([vdD98])

Every definable set $X \subset M^n$ can be triangulated, namely there is a finite (open) simplicial complex K and a definable homeomorphism $f : |K|^M \to X$.

In general two compact polyhedra $|K|$ and $|L|$ can be homeomorphic without being PL-homeomorphic. However:

Theorem (O-minimal Hauptvermutung: [Shi10])

Let K, L be finite closed simplicial complexes and let $f : |K|^M \to |L|^M$ be a definable homeomorphism. Then K and L have isomorphic subdivisions. So there is a PL-homeomorphism $g : |K|^M \cong |L|^M$.

Definable groups

Definition

A definable group is a definable set $G \subset M^n$ with a definable group operation.

Example

$$SO_2(M) = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a^2 + b^2 = 1 \right\}.$$

Example

A non singular elliptic curve $y^2 = x^3 + ax + b$ in $\mathbb{P}^2(M)$ ("real" case) or in $\mathbb{P}^2(M[\sqrt{-1}])$ ("complex" case).

Tori

Lie tori

Every connected compact abelian real Lie group is Lie-isomorphic to a torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.

In the o-minimal context $\mathbb{Z}$ does not exist, but we can define:

Definable tori

Let $\mathbb{T}^1(M) = [0,1) \subset M$ with the following group operation:

$$x * y = \begin{cases} x + y & \text{if } x + y < 1 \\ x + y - 1 & \text{otherwise} \end{cases}$$

Similarly we define the n-th torus $\mathbb{T}^n = [0,1)^n$.

t-topology

Theorem ([Pil88])

Any definable group $(G,*)$ admits a unique group topology, called the t-topology, such that G is a finite union $U_1 \cup \ldots \cup U_n$ with each U_i open in G and definably homeomorphic to M^n ($n = dim(G)$).

So when $M = (\mathbb{R}, <, \ldots)$, any definable group G is a real Lie group.

Example

With the t-topology the torus $\mathbb{T}^1 = [0,1)$ (with addition modulo 1) is definably homeomorphic to S^1 (the unit circle in M), so the t-topology does not coincide with the subspace topology $[0,1) \subset \mathbb{R}$.

t-topology

Definition

G is **definably compact** if every definable curve $f : (0, \varepsilon) \to G$ has a limit in G in the t-topology.
G is **definably connected** if G has no proper definable clopen subset in the t-topology. This is equivalent to say that G has not definable subgroups of finite index.

By "Robson's embedding theorem" we have:

Fact

Every definable group G can be embedded in some M^m, namely G is definably isomorphic to a group $G' \subset M^m$ such that the t-topology of G' coincides with the subspace topology inherited from M^m.

Such a G' is definably compact iff it is closed and bounded in M^m.

One-dimensional definable groups

We have seen various examples of one-dimensional definable groups: $SO(2, M)$, $\mathbb{T}^1(M)$, elliptic curves in $\mathbb{P}^2(M)$.

Note that $SO(2, \mathbb{R})$ is Lie-isomorphic to $\mathbb{T}^1(\mathbb{R})$ but not definably isomorphic in $(\mathbb{R}, <, +, \cdot)$ (i.e. the isomorphism is not semialgebraic).

Theorem ([Str94b])

Let G be a definably compact definably connected abelian definable group in M. Assume $dim(G) = 1$. Then G, with the t-topology, is definably homeomorphic to $\mathbb{T}^1$ (equivalently: to the unit circle S^1).

A natural conjecture is that when $dim(G) = n$, then G is definably homeomorphic to $\mathbb{T}^n$. The difficulty is that G may have no one-dimensional definable subgroups ([PS99]).

Main result

Theorem ([BB11])

Let G be a definably compact definably connected abelian n-dimensional definable group with $n \neq 4$. Then G is definably homeomorphic to $\mathbb{T}^n$. In the semialgebraic case the proviso $n \neq 4$ is not needed.

Main steps.

1. The result holds in the semialgebraic case;
2. Homotopy transfer;
3. $\pi_1(G) \cong \mathbb{Z}^n \cong \pi_1(\mathbb{T}^n)$ ([EO04]);
4. $\pi_k(G) = 0$ for $k > 0$;
   G is definably homotopy equivalent to $\mathbb{T}^n$ ([BMO10]);
5. If $dim(G) \neq 4$, there is a finite cover $f : G \to \mathbb{T}^n$ (as spaces). Use f to put a semialgebraic group operation on $dom(G)$.

□

Step 1: Semialgebraic case

Lemma (Elimination of parameters)

Let G be a semialgebraic group over a real closed field M. Then G is semialgebraic homeomorphic (but not necessarily isomorphic) to a semialgebraic group over $\overline{\mathbb{Q}}^{real} \prec M$.

Proof.

- We can assume that the t-topology of G is the topology inherited from the ambient space M^m (Robson's embedding theorem).
- By the triangulation theorem we can assume that $dom(G) = |K|$ where K is a finite simplicial complex definable without parameters.
- However the group operation may need parameters from M.
- Since $\overline{\mathbb{Q}}^{real} \prec M$, there is a possibly different commutative group operation $\oplus$ on $|K|$ which is defined over $\overline{\mathbb{Q}}^{real}$.

□

Step 1: Semialgebraic case

Theorem ([BB11])

Let G be a semialgebraic group of dimension n over a real closed field M. Suppose G is definably compact, definably connected, abelian. Then G is definably homeomorphic to $\mathbb{T}^n(M)$.

Proof.

- We can assume that G is defined over $\overline{\mathbb{Q}}^{real} \prec M$. Consider $G(\mathbb{R})$.
- $G(\mathbb{R})$ is a compact connected abelian Lie group with the t-topology.
- There is an analytic isomorphism $h : G(\mathbb{R}) \to \mathbb{T}^n(\mathbb{R})$.
  h is definable in the o-minimal structure $\mathbb{R}_{an}$.
- By Shiota's o-minimal Hauptvermutung there is a semialgebraic homeomorphism $f : G(\mathbb{R}) \to \mathbb{T}^n(\mathbb{R})$.
  Since $\overline{\mathbb{Q}}^{real} \prec M$ we can take f defined without parameters.
- The same formula gives a definable homeomorphism $f^M : G(M) \to \mathbb{T}^n(M)$.

Step 2: Homotopy transfer

Recall that homotopy equivalence is much weaker than homeomorphism.

Example

The figure "8" is homotopy equivalent to $\mathbb{R}^2$ minus two points.

The homotopy category is more flexible than the topological category.

Fact

- *$|K|$ and $|L|$ are homotopy equivalent if and only if they are semialgebraically homotopy equivalent.*
- *But two polyhedra $|K|$ and $|L|$ can be homeomorphic without being definably homeomorphic in any o-minimal expansion of $\mathbb{R}$.*

Step 2: Homotopy transfer

Let M be an o-minimal expansion of a field, and let X be a semialgebraic set defined over $\overline{\mathbb{Q}}^{real} \subset M$. Look at $X(\mathbb{R})$ and $X(M)$.

Theorem (Homotopy transfer)

1. $\pi_1{}^{def}(X(M)) \cong \pi_1(X(\mathbb{R}))$ *([BO02]);*
2. $H_*{}^{def}(X(M)) \cong H_*(X(\mathbb{R}))$ *([BO02]);*
3. $H^{*def}(X(M); \mathbb{Q}) \cong H^*(X(\mathbb{R}); \mathbb{Q})$ *([EO04] and duality);*
4. $\pi_n{}^{def}(X(M)) \cong \pi_n(X(\mathbb{R}))$ *([BO09]).*

The superscript "def" refers to the relativization to the definable category and will be omitted in the sequel.

Failure of topological transfer (see [BO03, Shi10, BB11])

If $|K|(M)$ is a definable manifold, then $|K|(\mathbb{R})$ is a (PL) manifold. But if $|K|(\mathbb{R})$ is a topological manifold, $|K|(M)$ need not be a definable manifold.

Step 2: Homotopy transfer

Proposition

If X is a definable set, $\pi_1(X)$ is finitely generated. Similarly $H_n(X)$ and $H^n(X)$ are finitely generated for all n.

Proof.

By homotopy transfer after triangulating X □

However $\pi_n(X)$ may not be finitely generated for $n > 1$.

Example

Let $X = S^1 \wedge S^2$. Then $\pi_2(X) = \mathbb{Z}^{(\omega)}$.

Indeed, for $n > 1$, $\pi_n(X) = \pi_n(\tilde{X})$ where $\tilde{X}$ is the universal cover of X. When $X = S^1 \wedge S^2$ the space $\tilde{X}$ is an infinite line with infinitely many copies S^2 attached to it, so clearly $\pi_2(\tilde{X})$ is not finitely generated as a group (however it is finitely generated as a $\mathbb{Z}[\pi_1(X)]$-module).

Step 3: Torsion

Theorem ([Str94a])

Let G be a definable abelian group. Then the k-torsion subgroup $G[k]$ is finite.

Question ([PS99])

Suppose G is definably compact. Is $G[k]$ non-empty?

Theorem ([EO04])

Let G be a definably compact definably connected abelian group of dimension n. Then $G[k] \cong \mathbb{T}^n[k]$.

The proof depends on the study of $\pi_1(G)$. One shows that $\pi_1(G) \cong \pi_1(\mathbb{T}^n)$ and that $G[k] \cong \pi_1(G)/k\pi_1(G)$. So we need to study $\pi_1(G)$.

Step 3: Fundamental group

Theorem ([EO04])

Let G be a definably compact definably connected abelian group of dimension n. Then $\pi_1(G) \cong \mathbb{Z}^n$.

Proof.

- Since $dom(G)$ is a definable set, $\pi_1(G)$ is finitely generated.
- Since G is abelian, the map $p_k : G \to G$, $x \mapsto kx$, is a homomorphisms.
- It is also a definable covering map, so it induces an injective homomorphism $p_{k*} : \pi_1(G) \to \pi_1(G)$, given by $[\gamma] \mapsto k[\gamma]$.
- Since this holds for every k, $\pi_1(G)$ is torsion free.
- Being also abelian and finitely generated, $\pi_1(G) \cong \mathbb{Z}^s$ for some s.
- The proof of $s = n$ uses the cup product in cohomology plus the theory of Hopf-spaces (proof omitted).

□

Step 4: Higher homotopy groups

We have seen that if X is a definable set $\pi_m(X)$ may not be finitely generated. However:

Lemma ([BMO10])

If G is a definably connected definable group in M, then $\pi_m(G)$ is finitely generated for all m.

Proof.

1. If $\mathbf{G}$ is a real Lie group, then the fundamental group $\pi_1(\mathbf{G})$ acts trivially on $\pi_m(\mathbf{G})$ for $m > 1$, namely $\mathbf{G}$ is a simple space.
2. (Serre 1953) If X is a simple space and $H_m(X)$ is finitely generated for all m, then $\pi_m(X)$ is finitely generated for all m.
3. Being a definable group, our group G is a simple space in the definable category.
4. Conclude by a suitable homotopy transfer.

□

Step 4: Higher homotopy groups

In the abelian case we get:

Corollary ([BMO10])

Let G be a definably connected definable abelian group in M. Then $\pi_m(G) = 0$ for all $m > 1$.

Proof.

- The morphism $p_k : G \to G$, $x \mapsto kx$, is a covering map, so it induces an injective endomorphism of $\pi_m(G)$ given by multiplication by k.
- Since $m > 1$ this is actually an automorphism of $\pi_m(G)$ [BO09].
- Since this holds for all k, we deduce that $\pi_m(G)$ is divisible.
- Since it is also abelian and finitely generated, it must be zero.

□

In particular this gives a proof that $\pi_m(\mathbb{T}^n) = 0$ for $m > 0$ without factoring $\mathbb{T}^n$ into one-dimensional subgroups.

Step 4: Higher homotopy groups

Theorem ([BMO10])

Let G be a definably connected definably compact definable abelian group in M. Then G is definably homotopy equivalent to $\mathbb{T}^n(M)$.

Proof.

- By [EO04], $\pi_1(G) \cong \mathbb{Z}^n$.
- Consider the map $f : \mathbb{T}^n \to G$ sending $(t_1, \ldots, t_n) \in [0,1)^n$ to $\gamma_1(t_1) + \ldots + \gamma_n(t_n)$ where $[\gamma_1], \ldots, [\gamma_n]$ are free generators of $\pi_1(G)$.
- Then clearly $f_* : \pi_1(\mathbb{T}^n) \cong \pi_1(G)$.
- Since $\pi_m(G) = 0$ for $m > 1$, f induces an isomorphism on all the π_m's.
- By the o-minimal version of Whitehead's theorem ([BO09]) f is a definable homotopy equivalence.

□

Step 5: Homotopy tori

We have seen that our group G is definably homotopy equivalent to $\mathbb{T}^n(M)$ and we want to prove that it is definably homeomorphic to it.

We could try to transfer "Borel's conjecture" from $\mathbb{R}$ to M:

Borel's conjecture

Let X be a PL-manifold homotopy equivalent to $\mathbb{T}^n(\mathbb{R})$ (considered as a *PL*-manifold under a standard triangulation). Then X is PL-homemorphic to $\mathbb{T}^n(\mathbb{R})$.

... but unfortunately this conjecture is false (it turns out that X is indeed homeomorphic to $\mathbb{T}^n(\mathbb{R})$ but the homomorphism is not necessarily PL).

Step 5: Homotopy tori

As a partial substitute for Borel's conjecture we have:

Theorem (Homotopy tori)

Let X be a (closed) PL-manifold of dimension $n \neq 4$ homotopy equivalent to $\mathbb{T}^n(\mathbb{R})$. Then there is a finite PL-covering $f : \mathbb{T}^n(\mathbb{R}) \to X$.

References

For $n \geq 5$, see [HW69]. The case $n = 3$ follows from results in [KS77] plus the positive solution of Poincaré's conjecture. For $n \leq 2$ X is already PL-homeomorphic to $\mathbb{T}^n(\mathbb{R})$.

Step 5: Homotopy tori

By triangulating our group G and transfering from $\mathbb{R}$ to M the result on Homotopy tori we get:

Lemma

There is a semialgebraic (even PL) finite cover

$$f : \mathbb{T}^n(M) \to G(M)$$

(as spaces, not as groups).

This is nice, but it would have been more useful to have a cover going the other way around.

Step 5: Homotopy tori

Lemma

Let G be definably compact, definably connected, abelian, of dim $\neq 4$. Then there is a definable finite cover $h : G(M) \to \mathbb{T}^n(M)$.

Proof.

- Start with the finite cover $f : \mathbb{T}^n(M) \to G(M)$ (as spaces).
- Lift the group operation on G to a definable group operation $*$ on $\mathbb{T}^n$. So f becomes a cover of definable groups $f : H = (\mathbb{T}^n, *) \to G$.
- Since $\ker(f) < H$ is finite, $\ker(f) < H[m]$ for some m.
- So we get a group cover $G \cong H/ker(f) \to H/H[m]$.
- H is a definable abelian group, so by [Str94a] $H[m]$ is finite. Deduce that $H/H[m] \cong H$ (as H is definably connected).
- So we get a definable cover $h : G(M) \to \mathbb{T}^n(M)$.

□

Step 5: Reduction to the semialgebraic case

Lemma

Let G be definably compact, definably connected, abelian, of dim $\neq 4$. Then G is definably homeomorphic (not isomorphic!), to a semialgebraic abelian group.

Proof.

- We can assume that $dom(G) = |K|$ (triangulation).
- We have a definable cover $h : G \to \mathbb{T}^n$ (as spaces).
- Since $\mathbb{T}^n$ is semialgebraic, there is a semialgebraic cover $h' : G' \to \mathbb{T}^n$ and a definable homeomorphism $\phi : G \approx G'$ with $h' \circ \phi = h$. (see [EJP10])
- By the o-minimal Hauptvermutung there is a semialgebraic homeomorphism $\psi : G \approx G'$ and therefore $h' \circ \psi : G \to \mathbb{T}^n$ is a semialgebraic cover.
- $\mathbb{T}^n$ admits a semialgebraic commutative group operation which can be lifted to G via the cover.

□

Conclusions

By a reduction to the semialgebraic case we have proved that a definably connected definably compact definable abelian group G with $dim(G) \neq 4$ is definably homeomorphic to $\mathbb{T}^n(M)$.

When $dim(G) = 4$ the same proof gives the following weaker result:

Corollary

Suppose $dom(G) = |K|(M)$. Then $|K|(\mathbb{R})$ is homeomorphic to $\mathbb{T}^n(\mathbb{R})$.

Proof.

By results in [FQ90] a PL-manifold homotopy equivalent to $\mathbb{T}^n(\mathbb{R})$ is homeomorphic to $\mathbb{T}^n(\mathbb{R})$. □

However in principle the homeomorphism may be wild (not PL), so we cannot deduce $|K|(M)$ is definably homeomorphic to $\mathbb{T}^n(M)$.

Conclusions

A possible approach to deal with the case $dim(G) = 4$ is to replace G with $G \times \mathbb{T}^1$ (a group of $dim = 5$). But ...

Question

Let X be a semialgebraic set and suppose that $X(\mathbb{R}) \times \mathbb{T}^1(\mathbb{R})$ is semialgebraically homeomorphic to $\mathbb{T}^5(\mathbb{R})$. Is $X(\mathbb{R})$ semialgebraically homeomorphic to $\mathbb{T}^4(\mathbb{R})$?

The answer is yes if $\mathbb{T}^1$ is replaced by $\mathbb{R}$ (Shiota). With $\mathbb{T}^1$ there could be problems.

Conclusions

Another possibility to deal with the dim = 4 case is to use the work of many authors on the "infinitesimal subgroup" G^{00} of G (see [Pil04, BOPP05]):

Theorem

1. *If G is a definable group in M and $M^* \succ M$ is saturated, then $\mathbf{G} = G(M^*)/G^{00}(M^*)$ is a compact real Lie group [BOPP05].*
2. *When G is definably compact, G^{00} is torsion free and $dim_M(G) = dim_{\mathbb{R}}(\mathbf{G})$[HPP07].*

When G is definably compact, definably connected and abelian, it follows that $\mathbf{G} \cong \mathbb{T}^n(\mathbb{R})$. We could try to transfer results from $\mathbf{G}$ to G and deduce that G is definably homeomorphic to $\mathbb{T}^n(M)$.

Conclusions

Theorem (Transfer from G to G)

Let G be definably compact and let $\mathbf{p} : G \to \mathbf{G} = G(M^*)/G^{00}(M^*)$ be the projection.

1. *$dim_M(G) = dim_{\mathbb{R}}(\mathbf{G})$ [HPP07].*
2. *The image under $\mathbf{p}$ of a nowhere dense definable $X \subset G$, has measure zero in $\mathbf{G}$. [HP09].*
3. *If $U \subset \mathbf{G}$ is open and $V \subset G$ is the preimage of U, then $\pi_1(U) \cong \pi_1^{def}(V)$ [BM11].*
4. *$\mathbf{G}$ determines the definable homotopy type of G [Bar09, BM11].*

Taking G abelian and $U = \mathbf{G}$ in (3) we obtain $\pi_1(G) \cong \pi_1(\mathbf{G})$ ($\cong \mathbb{Z}^n$). However (3) uses (2), which in turn uses $\pi_1(G) \cong \mathbb{Z}^n$.

Conjecture

If G is triangulated by $|K|(M)$, then $\mathbf{G}$ can be triangulated by $|K|(\mathbb{R})$.

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