Motivic integration on Artin *n*-stacks

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Prestacks

(This treatment of stacks is due to B. Toën and G. Vezzosi.)
Let S be a fixed base scheme. Let $(\mathrm{Aff}\,/S)$ be the category of affine schemes over S with the etale topology.

Let **Sset** denote the model category of simplicial sets. Let $(\mathrm{Aff}\,/S)^\wedge$ denote the category $\mathbf{Sset}^{(\mathrm{Aff}\,/S^{op})}$ (the category of preheaves on $(\mathrm{Aff}\,/S)$ taking values in **Sset**).

$(\mathrm{Aff}\,/S)^\wedge$ is a model category in which the weak equivalences and fibrations are the objectwise weak equivalences and objectwise fibrations respectively. This is the *model category of prestacks over* S.

The simplicial Yoneda lemma gives an embedding

$$(\mathrm{Aff}\,/S) \to (\mathrm{Aff}\,/S)^\wedge.$$

Homotopy group sheaves

Let F be a prestack over S.

Let $\pi_0(F)$ be the sheaf associated with the presheaf

$$\pi_0^{pr}(F) : U \to \pi_0(F(U)).$$

For any affine scheme U, any morphism $u : U \to F$ and any integer $i > 0$, the sheaf $\pi_i(F, u)$ on $(\mathrm{Aff}\,/U)$ is the sheaf associated with the presheaf

$$\pi_i^{pr}(F, u) : V \to \pi_i(F(V), u|_V).$$

Stacks

A morphism $F \to G$ is said to be a *local equivalence* if it induces an isomophism of π_i sheaves for $i \geq 0$.

The *model category of stacks over* S is the left Bousfield localization of $(\mathrm{Aff}\,/S)^\wedge$ with respect to local equivalences. We denote this category by $(\mathrm{Aff}\,/S)^{\sim,et}$.

The *category* $St(S)$ *of stacks over* S is the homotopy category $Ho((\mathrm{Aff}\,/S)^{\sim,et})$.
We say that a stack F is *n-truncated* or that it is an *n-stack* if its π_i-sheaves are trivial for $i > n$.
e.g. A stack is a 0-stack iff it is a sheaf.

Monomorphisms and epimorphisms

- A morphism $F \to G$ of stacks is a monomorphism if it induces a monomorphism $\pi_0(F) \to \pi_0(F)$ and an isomorphism of π_i sheaves for $i > 0$.
- A morphism $F \to G$ of stacks is an epimorphism if $\pi_0(F) \to \pi_0(G)$ is an epimorphism of sheaves.

Geometric stacks

The notions of *k-geometric stack, k-representable morphism and k-smooth maps* are defined by induction on k for $k \geq -1$:

- Suppose that the notion of k-geometric stack has been defined. Then a morphism $F \longrightarrow G$ of stacks is k-representable if for any affine scheme T and any morphism $T \longrightarrow G$, the stack $F \times^h_G T$ is a k-geometric stack.
- Suppose the notion of k-representable morphism has been defined and the notion of l-smoothness has been defined for $l < k$. Let $F \longrightarrow G$ be a k-representable morphism of stacks. We say that this morphism is k-smooth if for any affine scheme T and any morphism $T \longrightarrow G$, there exists a disjoint union of affine schemes $U = \coprod_i U_i$ and a $(k-1)$-smooth epimorphism $U \longrightarrow F \times^h_G T$ such that $U_i \longrightarrow T$ is smooth for all i.

Geometric stacks (contd.)

- A stack is (-1)-geometric if it is an affine scheme.
- Suppose that the notion l-geometric stack has been defined for $l < k$. Then a stack F is k-geometric if it satisfies the following two conditions:
    1. $F \to F \times^h F$ is $(k-1)$-representable.
    2. There exists a disjoint union of affine schemes $U = \coprod_i U_i$ and a $(k-1)$-smooth epimorphism $U \to F$. (We say that $U \to F$ is a *smooth atlas* of F.)

An *Artin stack* is a stack that is k-geometric for some k.

Examples:

- Algebraic spaces are geometric.
- Suppose G is an flat abelian group scheme. Consider the stack defined by the simplicial presheaf

$$U \to K(G(U), n)$$

where $K(G(U), n)$ denotes the n-th Eilenberg-Maclane space of $G(U)$, i.e.

$$\pi_0(K(G(U), n)) = \{*\}$$

and

$$\pi_i(K(G(U), n), *) = \begin{cases} 0 & \text{for all } i \neq n, \\ G(U) & \text{for } i = n. \end{cases}$$

We denote this stack by $K(G, n)$. It can be proved that if G is a group scheme, then $K(G, n)$ is geometric.

Properties of geometric stacks

- If F is k-geometric, U is an affine scheme any morphism $U \to F$ is $(k-1)$-representable. Thus if U and V are affine schemes and $U \to F$ and $V \to F$ are any morphisms, $U \times_F^h V$ is $(k-1)$-geometric.
- A k-geometric stack is a $(k+1)$-stack.
- Suppose F is an k-geometric stack and $f : F \to G$ is a surjective $(k-1)$-representable morphism. Then G is k-geometric.

Extending geometrical notions to Artin stacks

Many of the standard notions related to schemes can be extended to Artin stacks. We omit the explicit definitions and merely note that these extensions are possible.

- Properties of schemes which are local with respect to the smooth topology can be defined for Artin stacks. (Locally noetherian, reduced, normal, etc.)
- Properties of morphisms of schemes which are stable with with respect to the base change and local with respect to the smooth topology can be defined for morphisms of Artin stacks.

Extending geometrical notions (contd.)

- Given any stack X, we can define the set $|X|$ of points on X. If X is an Artin stack, $|X|$ comes equipped with a natural topology called the Zariski topology in which the closed sets are of the form $|Y|$ where Y is a closed substack of X.
  Fact: If $X \to Y$ is universally open, the morphism $|X| \to |Y|$ is open with respect to the Zariski topology.
  Thus the if X is an Artin stack and $U \to X$ is an atlas, the topology on $|X|$ can be understood in terms of the topology of $|U|$. Thus the usual properties of constructible subsets of schemes (Chevalley's theorem, etc.) can be generalized to Artin stacks.

Strongly quasi-compact stacks

We say that an Artin stack is *quasi-compact* if there exists an epimorphism $U \to X$ where U is an affine scheme.
We will also need a stronger version of quasi-compactness to ensure that loop stacks are quasi-compact.
We define the notion of *strong quasi-compactness* for k-geometric stacks and k-representable morphisms by induction on k:

- Suppose that the notion of a strongly quasi-compact k-geometric stack has been defined. Then a k-representable morphism $F \to G$ is strongly quasi-compact if for any affine scheme T and any morphism $T \to G$, the stack $F \times_G^h T$ is strongly quasi-compact.
- Suppose that the notion of a strongly quasi-compact l-geometric stack has been defined for all $l < k$. Then a k-geometric stack X is strongly quasi-compact if:
  1. The morphism $X \to X \times^h X$ is strongly quasi-compact.
  2. X is quasi-compact.

Strongly quasi-compact stacks (contd.)

The main reason why we need this notion is as follows:
If X is strongly quasi-compact, U is an affine scheme and $u : U \to X$ is any morphism, then the stack $U \times^h_X U$ is strongly quasi-compact.
Remark: The loop stack of X at u is given by the restriction of the stack

$$(U \times^h_X U) \times^h_{(U \times U)} U.$$

to $(\mathrm{Aff}\,/U)$.

If a stack is strongly quasi-compact and locally finitely presented, we say that it is *strongly finitely presented*.
If X is a strongly finitely presented stack over a noetherian scheme, $|X|$ is a noetherian topological space.

Weil restriction

Let $S \to T$ be a morphism of schemes. Let

$$f : (\mathrm{Aff}\,/T) \to (\mathrm{Aff}\,/S)$$

be the functor $U \to U \times_T S$.
The *Weil restriction functor* corresponding to $S \to T$ is the pullback functor $f_* : Pr(\mathrm{Aff}\,/S) \to Pr(\mathrm{Aff}\,/T)$ associated with f.

$$f_* F(U) = F(f(U)).$$

f takes etale covers to etale covers and preserves fiber products.
Thus f_* takes sheaves to sheaves.

Weil restriction (contd.)

Similarly, f induces a morphism $f_* : (\mathrm{Aff}\,/S)^\wedge \to (\mathrm{Aff}\,/T)^\wedge$.
Fact: f_* preserves local equivalences. Thus f_* induces a functor $St(S) \to St(T)$.

We recall the following well-known theorem of Greenberg.

Theorem

Let k be a field and let R be a local Artinian ring with residue field k. Let f denote the functor

$$f : (\mathrm{Aff}\,/\,\mathrm{Spec}(k)) \to (\mathrm{Aff}\,/\,\mathrm{Spec}(R))$$

defined as $U \to U \times_{\mathrm{Spec}(k)} \mathrm{Spec}(R)$. Then if X is a scheme of finite type over $\mathrm{Spec}(R)$, the sheaf $f_*(X)$ is represented by a scheme of finite type over k.

Conventions

- k will denote a field of characteristic zero.
- $R_n := k[[t]]/(t^{n+1})$ for all $n \in \mathbb{Z}_{\geq 0}$.
- $R_\infty = R := k[[t]]$.
- $\mathbb{D}_n = \mathrm{Spec}(R_n)$ for all $n \in \mathbb{Z}_{\geq 0}$.
- $\mathbb{D}_\infty := \mathbb{D} = \mathrm{Spec}(R)$.
- For all $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, σ_n is the morphism $\mathbb{D}_n \to \mathbb{D}_0$.
- All Artin stacks (in particular schemes) over $\mathbb{D}$ are assumed to be flat, reduced, strongly finitely presented and of pure dimension over $\mathbb{D}$.

Arc spaces

Let X be a scheme over $\mathbb{D}$.
For $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, the *space of n-arcs* (or simply *arcs* if $n = \infty$) on X is the sheaf

$$Gr_n(X) = (\sigma_n)_*(X \times_{\mathbb{D}} \mathbb{D}_n).$$

(When $n = \infty$, we simply write $Gr(X)$ instead of $Gr_\infty(X)$.)

By Greenberg's theorem, if $n < \infty$, the sheaf $Gr_n(X)$ is a scheme of finite type over k.

The elements of $|Gr_n(X)|$ are called *n-arcs (or simply arcs if $n = \infty$) on X*.

Truncation maps

For $m, n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ with $m > n$, let $\tau_n^m : Gr_m(X) \to Gr_n(X)$ denote the morphism induced by $\mathbb{D}_n \to \mathbb{D}_m$. (If $m = \infty$, we write τ_n instead of τ_n^m.)
If $n < m < \infty$, the morphism τ_n^m is an affine map. Thus

$$Gr(X) := \varprojlim Gr_n(X)$$

is a scheme over k (not of finite type).

For $n < m < \infty$, ρ_n^m denotes the function $(\tau_n^m)|_{\tau_m(|Gr(X)|)}$.

Truncation maps on smooth schemes

Let X be a smooth scheme over $\mathbb{D}$. Then by Hensel's lemma, τ_n^m is surjective.
Suppose X is the complete intersection of an m-tuple of polynomials $f := (f_1, \ldots, f_m)$ in $R[x_1, \ldots, x_{m+d}]$.
Let $a = (a_1(t), \ldots, a_{m+d}(t))$ an arc on X. We would like to find all vectors $u \in k^{m+d}$ such that

$$a(t) + ut^{n+1} \equiv b(t) \pmod{t^{n+2}}$$

for some arc b on X. (In other words, we want to find $(\rho_n^{n+1})^{-1}(a)$.)

Truncation maps on smooth schemes (contd.)

For this, we need to solve the congruence

$$f(a + ut^{n+1}) \equiv D(\overline{f})(a(0)) \cdot (u)t^{n+1} \equiv 0 (\text{mod } t^{n+2})$$

where $\overline{f}$ is the reduction of f modulo t. $D(\overline{f})$ has rank m. Thus the set of all possible u forms an d-dimensional subspace of k^{m+d}. This result continues to hold on singular schemes "away from the singular locus".

Singular locus

Let X a scheme over $\mathbb{D}$ with $\dim(X/\mathbb{D}) = d$. Let $\mathcal{J}_{X/\mathbb{D}}$ denote the d-th Fitting ideal of $\Omega_{X/\mathbb{D}}$. Let X_{sing} denote the closed subscheme of X cut out by $\mathcal{J}_{X/\mathbb{D}}$. We call X_{sing} the *singular locus of X*.

Let $Gr_n^{(e)}(X)$ denote the space of n-arcs having intersection number $\leq e$ with X_{sing}. Then $Gr_n^{(e)}(X)$ is an open subscheme of $Gr_n(X)$.

Truncation maps on singular schemes

Let X be a scheme over $\mathbb{D}$ (not necessarily smooth) with $\dim(X/\mathbb{D}) = d$.

Lemma

Let $n \geq e$. Then $\rho_n^{n+1} : \tau_{n+1}(|Gr^{(e)}(X)|) \to \tau_n(|Gr^{(e)}(X)|)$ is an $\mathbb{A}^d$-bundle.

Indeed, if a point of $\tau_n(|Gr^{(e)}(X)|)$ is represented by $x : \mathbb{D} \to X$, the fibre of ρ_n^{n+1} over this point is an affine space whose translation space is $Hom_R(x^*\Omega_{X/\mathbb{D}}, R) \otimes_R (t^n R/t^{n+1}R)$ which can be checked to be a d-dimensional k-vector space.
(Ref: E. Looijenga, *Motivic measures*)

Weakly stable and stable sets

Definition

Let A be a subset of $|Gr(X)|$. We say that A is *weakly stable at level n* if $\tau_n(A)$ is a constructible set and

$$A = \tau_n^{-1}(\tau_n(A)).$$

We say that A is *weakly stable* if it is weakly stable at level n for some integer n.

Definition

Let A be a subset of $|Gr(X)|$. We say that A is *stable at level n* if:

1. A is weakly stable at level n.
2. $\tau_{m+1}(A) \to \tau_m(A)$ is a $\mathbb{A}^d$-bundle for $m \geq n$.

Thus, any weakly stable subset of $|Gr^{(e)}(X)|$ is stable.

Grothendieck ring of varieties over k

Motivic measure is a measure on $|Gr(X)|$ taking values in the Grothendieck ring of varieties over k. This ring is defined as follows:

Definition

Let $K_0(Var/k)$ denote the free abelian group generated by symbols of the form $[V]$ where V is a variety over k, subject to the following relations:

1. If V and V' are isomorphic, then
$$[V] = [V'].$$
2. If V is a variety over k and W is a closed subvariety, then
$$[V] = [W] + [V/W].$$

This is the *Grothendieck group of varieties over k*.

Grothendieck ring of varieties over k (contd.)

We define a product on this group by the formula $[V_1] \cdot [V_2] = [V_1 \times_k V_2]$. This gives $K_0(Var/k)$ the structure of a commutative ring. This is called the *Grothendieck ring of varieties over k*.

If $S \to T$ is a morphism of schemes which is a Zariski-locally trivial S_0 fibration for some scheme S_0, then clearly $[S] = [S_0] \cdot [T]$.

If S is a scheme and C is a constructible subset of $|S|$, we define $[C] = \sum_{i=1}^{p} [S_i]$ where $C = \coprod_i |S_i|$ is a decomposition of C into locally closed subsets of $|S|$.

Notation: $\mathbb{L} := [\mathbb{A}_k^1]$.

Motivic measure on stable sets

If A is a subset of $|Gr(X)|$ which is stable at level n, then in the Grothendieck ring of varieties over k, we obtain the relation

$$[\tau_{m+1}(A)] = \mathbb{L}^d \cdot [\tau_m(A)]$$

for all $m \geq n$.
Thus the element $[\tau_m(A)] \cdot \mathbb{L}^{-md} \in K_0(Var/k)[\mathbb{L}^{-1}]$ is independent of m when m is sufficiently large.
Let $M_k := K_0(Var/k)[\mathbb{L}^{-1}]$.
We define

$$\mu : \{\text{Stable subsets of } |Gr(X)|\} \to M_k$$

by the formula

$$\mu(A) = [\tau_m(A)] \cdot \mathbb{L}^{-md}$$

for sufficiently large m.

Language of Denef-Pas

A *language of Denef-Pas* is a three-sorted language of the form

$$\mathcal{L}_{DP} = (\mathcal{L}_{Val}, \mathcal{L}_{Res}, \mathcal{L}_{Pres}, \mathrm{ord}, \overline{\mathrm{ac}})$$

with three sorts - Val-sort (for valued fields), Res-sort (for residue field) and Ord-sort (for ordered groups). The languages $\mathcal{L}_{Val}$, $\mathcal{L}_{Res}$ and $\mathcal{L}_{Pres}$ used for these sorts are defined as follows:

– $\mathcal{L}_{Pres}$: The *Presburger language* is an expansion of the language of ordered groups.

$$\mathcal{L}_{Pres} = \{+, -, 0, 1, \leq\} \cup \{\equiv_n | n \in \mathbb{N}, n > 1\}$$

where $\equiv_n$ denotes the equivalence relation module n.

Language of Denef-Pas (contd.)

- $\mathcal{L}_{Res}$ is an expansion of the language $(+, -, \cdot, 0, 1)$ of rings.
- $\mathcal{L}_{Val}$ is the language of rings.

For any field K, we consider $(K((t)), K, \mathbb{Z})$ as a structure for of $\mathcal{L}_{DP}$ where $\mathrm{ord} = \mathrm{ord}_t$, $\overline{\mathrm{ac}}(a(t))$ is the *angular component* of $a(t)$ defined as the non-zero coefficient of lowest degree if $a(t)$ is non-zero and $\overline{\mathrm{ac}}(0) = 0$.

Definition

A subset of $|Gr(X)|$ is said to be *definable* if it is locally given by a formula in the language $\mathcal{L}_{DP} \cup k[[t]]$.

Properties of definable sets

- If C is a definable set, $\tau_n(C)$ is a constructible subset of $|Gr_n(X)|$ for all n.
- If $f : X \to Y$ is a morphism of schemes over $\mathbb{D}$ and A is a definable subset of $|Gr(X)|$, then $f(C)$ is a definable subset of $|Gr(Y)|$.
- If $f : X \to Y$ is a morphism of schemes over $\mathbb{D}$ and C is a definable subset of $|Gr(Y)|$, then $f^{-1}(C)$ is a definable subset of $|Gr(X)|$.

A filtration on M_k

We wish to extend μ to the algebra of definable subsets of $|Gr(X)|$. In order to do so, we need to modify M_k.
For any $[V]/\mathbb{L}^m \in M_k$, define

$$\dim([V]/\mathbb{L}^m) = \dim(V) - m.$$

Let

$$F^m(M_k) := \{v \in M_k | \dim(v) \leq -m\}.$$

Then F^m defines a decreasing filtration on M_k. Let $\widehat{M_k}$ denote the completion of M_k with respect to the filtration F^*.

Decomposition into weakly stable sets

The following argument by Denef and Loeser allows us to extend μ to definable sets:

Lemma

Let X be a flat, reduced, scheme of finite type and pure dimension d over $\mathbb{D}$. Let C be a definable subset of $Gr(X)$. Then there exists a closed subscheme of $Y \subset X$ with $\dim(Y/\mathbb{D}) < \dim(X/\mathbb{D})$ and a countable family of disjoint definable subsets $\{C_i\}_{i \in \mathbb{Z}_{\geq 0}}$ of $|Gr(X)|$ such that:

1. *For each i, C_i is weakly stable at level n_i for some integer n_i.*
2. $C \backslash Gr(Y) = \bigcup_i C_i$.
3. $\lim_{i \to \infty} \dim(\tau_{n_i}(C_i)) - n_i d = -\infty$.

Decomposition into stable sets

We note that in this lemma, the C_i can actually be chosen to be *stable!*
Indeed, if C_i is weakly stable at level n_i, the set $C_i \cap |Gr^e(X)|$ is stable at level $\max(n_i, e)$. Applying this argument to each C_i, we obtain a decomposition of $C \backslash Gr(Y \cup X_{sing})$ into stable sets.

Motivic measure for definable sets (contd.)

Thus if C is a definable subset of $Gr(X)$, we may define

$$\mu(A) = \sum_i \mu(C_i)$$

where the C_i are stable, chosen as above. The sum on the right converges in $\widehat{M_k}$ since

$$\lim_{i\to\infty} \dim(\tau_{n_i}(C_i)) - n_i d = -\infty.$$

It can be checked that μ satisfies the properties expected from a measure function. (Denef-Loeser)

Greenberg functor for stacks

As in the case of schemes, if X is an Artin stack over $\mathbb{D}$, we define

$$Gr_n(X) = (\sigma_n)_*(X \times_{\mathbb{D}} \mathbb{D}_n).$$

Some results:

- Gr_n preserves epimorphisms of stacks for all $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$.
- Gr_n takes strongly finitely presented m-geometric stacks over $\mathbb{D}$ to strongly finitely presented m-geometric stacks over $\mathbb{D}_0$.

Truncations of homotopy group sheaves

Let X be an Artin stack over $\mathbb{D}$. Let T be any affine scheme over $\mathbb{D}_0$ and $\widetilde{t} : T \times_{\mathbb{D}_0} \mathbb{D} \to X$ be any morphism. For each $n \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$, let $t_n : T \to Gr_n(X)$ be the morphism induced by $\widetilde{t}$.

Easy observation:

We have isomorphisms:

1. $\pi_0(Gr_n(X)) = Gr_n(\pi_0(X))$.
2. $\pi_i(Gr_n(X), t_n) = Gr_n(\pi_i(X, \widetilde{t}))$. (This map is actually a surjection when $char(k) = 0$.)

Definable sets of arcs on Artin stacks

Definition

Let X be an Artin stack over $\mathbb{D}$. Let $p : X' \to X$ be an atlas for X. We say that a set of arcs $C \in |Gr(X)|$ is *definable with respect to* p if $p^{-1}(C)$ is definable.

Easy results:

- If a subset of $|Gr(X)|$ is definable with respect to an atlas of X, it is definable with respect to any atlas of X. Thus we simply say that such a set is *definable*.
- If C Is a definable subset of $|Gr(X)|$, $\tau_n(C)$ is constructible for all n.
- If $f : X \to Y$ is a morphism of Artin stacks over $\mathbb{D}$ and C is a definable subset of $|Gr(X)|$, then $f(C)$ is a definable subset of $|Gr(Y)|$.
- If $f : X \to Y$ is a morphism of Artin stacks over $\mathbb{D}$ and C is a definable subset of $|Gr(Y)|$, then $f^{-1}(C)$ is a definable subset of $|Gr(X)|$.

Weakly stable definable sets

Weakly stable sets of arcs on Artin stacks are defined in the same manner as on schemes.

Let $f : X \to Y$ be a smooth morphism of Artin stacks. Let C be a weakly stable subset of $|Gr(Y)|$. Then $f^{-1}(C)$ is weakly stable. (This is a consequence of Hensel's lemma. First prove this under the assumption that X and Y are schemes and then prove for k-geometric stacks by induction on k.)

Thus a set of arcs on an Artin stack is weakly stable if and only if its preimage on some atlas is weakly stable.

Decomposition into weakly stable sets

Lemma

Let X be an Artin stack over $\mathbb{D}$ and let C be a definable set of arcs on X. Then there exists a closed substack X^C of X such that $\dim(X/\mathbb{D}) > \dim(X^C/\mathbb{D})$ *and:*

1. $C \backslash Gr(X^C)$ *is the disjoint union of a countable collection of weakly stable sets.*
2. X^C *is the minimal closed substack with respect to (1).*

Sketch of proof: Note that if X is a scheme, we already know the existence of some X^C satisfying (1). Take the intersection of all such subschemes to prove (2). Use the minimality of X^C to prove that this construction is local with respect to the smooth topology on X.

Very stable sets

Let X be an Artin stack over $\mathbb{D}$ with $\dim(X/\mathbb{D}) = d$. Let Z be a locally closed subset of $\tau_n(Gr(X))$. Let F be the reduced, locally closed substack of $Gr_n(X)$ such that $|F| = Z$. Let $Z' = \rho_n^{n+1}(Z)$. We say that Z is *very stable* if:

1. Z' is a locally closed subset of $|Gr_{n+1}(X)|$. Let F' be the reduced, locally closed substack of $Gr_{n+1}(X)$ such that $|F'| = Z'$.
2. Let T be any affine scheme over $\mathbb{D}_0$ and let $t' : T \to F'$ be any morphism. Let $t = \tau_n^{n+1} \circ t'$ and let $i > 0$. Then

$$\pi_i(F', t') \to \pi_i(F, t)$$

   is a surjective homomorphism of group sheaves, the kernel of which is Zariski-locally isomorphic (over T) to $\mathbb{G}_a^{d_i}$ for some integer d_i independent of t'.

Very stable sets (contd.)

3. $\pi_0(F') \to \pi_0(F)$ is a Zariski-locally trivial $\mathbb{A}^{d_0}$-bundle over $\pi_0(F)$ where

$$\sum_{i=0}^{\infty}(-1)^i d_i = d.$$

Note: In (3), the sheaves $\pi_0(F)$ and $\pi_0(F')$ are *not* assumed to be representable. We mean that if $Z \to \pi_0(F)$ is any morphism from a scheme Z, then

$$\pi_0(F') \times_{\pi_0(F)} Z \to Z$$

is a $\mathbb{A}^{d_0}$-bundle.

Stratification by very stable sets

Proposition

Let $n \geq e$ be integers. Let X be an Artin stack over $\mathbb{D}$ with $\dim(X/\mathbb{D}) = d$*. Let Z be a locally closed subset of $|Gr_n(X)|$ contained in $\tau_n(|Gr^{(e)}(X)|)$. Then Z has a stratification*

$$Z := Z_0 \supset Z_1 \supset \ldots \supset Z_k$$

by subsets Z_i which are closed with respect to Z, such that for each i, $Z_i \backslash Z_{i+1}$ is very stable.

Sketch of proof

Let F be a locally closed substack of $Gr_n(X)$ such that $|F| = Z$. Let $V \to X$ be a smooth atlas of X with V being an affine scheme.

Let U be the preimage of F in $Gr_n(V)$. Let $V_1 = V \times_X^h V$ and let $U_1 = U \times_F^h U$. Then U (resp. U_1) is locally closed in $Gr_n(V)$ (resp. $Gr_n(V_1)$) and is contained in $\tau_n(Gr^{(e)}(V))$ (resp. $\tau_n(Gr^{(e)}(V_1))$).

Let U' and U'_1 be the reduced, locally closed substacks of $Gr_{n+1}(V)$ and $Gr_{n+1}(V_1)$ such that

$$|U'| = (\rho_n^{n+1})^{(-1)}(|U|)$$

and

$$|U'_1| = (\rho_n^{n+1})^{(-1)}(|U_1|)$$

We know that $U' \to U$ is a $\mathbb{A}^p$-bundle for some p.

Sketch of proof (contd.)

Suppose X is m-geometric. The proof is by induction on m.
The case $m = -1$ is already known.
Suppose that the result has been proved for $m = k$ and assume that $m = k + 1$. Then V_1 is k-geometric and thus we may apply the induction hypothesis to $|U_1|$ to find an open substack W_1 of U_1 such that $|W_1|$ very stable. Choose W_1 to be maximal with respect to this property.
Step 1: Use the maximality of W_1 to prove that if G is the open substack of F which is the image of $W_1 \to F$, then $W_1 = W \times^h_G W$ where $W = U \times^h_G F$. Thus we may assume that $|U_1|$ is very stable. Suppose that $\pi_0(U_1') \to \pi_0(U)$ is an $\mathbb{A}^q$-bundle.
Step 2: The map $U_1' \to U' \times U'$ is a map between affine bundles over $U_1 \to U \times U$. This is an affine map on each fiber. Show that we can replace F by an open substack to reduce to the case in which the rank of this affine map is independent of the fiber. Denote this rank by ρ.

Sketch of proof (contd.)

Step 3: Check that $\pi_1(F', -) \to \pi_1(F, -)$ has a kernel which is locally isomorphic to $\mathbb{G}_a^{d_1}$ where $d_1 = q - \rho$. Use the fact that $|U_1|$ is very stable, along with the long exact sequence of homotopy group sheaves corresponding to the fiber sequence $U_1 \to U \to F$, to deduce that $|F|$ is very stable.
Step 4: Check that $\pi_0(F') \to \pi_0(F)$ is a $\mathbb{A}^{d_0}$-bundle where $d_0 = 2p - (q - d_1)$. Check that $\sum_i (-1)^i d_i = d$.

This completes the proof.

Grothendieck ring of Artin stacks (due to Toën)
The Grothendieck ring of strongly finitely presented Artin stacks is defined in a manner similar to the Grothendieck ring of varieties with the following additional relation:
Let F_0 be an affine scheme or a stack of the form $K(\mathbb{G}_a, n)$ for some integer $n > 0$. If $F \to F'$ is a Zariski-locally trivial F_0-fibration, then we have

$$[F] = [F' \times F_0].$$

An important consequence of this relation is:

$$[K(\mathbb{G}_a, n)] = \mathbb{L}^{(-1)^n}$$

for all $n > 0$.

Stable sets

It follows that if $Z = |F| \subset \tau_n(Gr(X))$ is a very stable set, and $(\rho_n^{n+1})^{-1}(Z) = Z' = |F'|$, then

$$[F'] = [F] \cdot \mathbb{L}^d.$$

Indeed, let

$$\dots F'_{p+1} \to F'_p \to \dots \to F'_1 \to F'_0 \to F$$

be the Postnikov system of $F' \to F$. Then if X is an m-stack, $F'_{m+k} = F'_m = F'$ for all $k \geq 0$. Also, $F'_{p+1} \to F'_p$ is a $K(\mathbb{G}_a^{d_{p+1}}, p+1)$ fibration. Since $\sum_i (-1)^i d_i = d$, the result follows immediately.
This motivates the following definition:

Stable sets (contd.)

Definition

Let X be an Artin stack over $\mathbb{D}$ with $\dim(X/\mathbb{D}) = d$. A definable set of arcs Z on X is said to be stable at level n if it is weakly stable at level n and if $\tau_m(Z)$ is a disjoint union of very stable sets forall $m \geq n$. We say that Z is stable if it is stable at level n for some integer n.

Having defined the notion of stable sets of arcs for Artin stacks, we can construct a motivic measure on the definable subsets of $|Gr(X)|$ by imitating the construction for motivic measure on schemes.

Change of variables formula

We say that a morphism of Artin stacks is 0-truncated if it induces an isomorphism of π_i sheaves for $i > 0$. We have the following "change of variables" formula for such morphisms.

Theorem

Let $h : Y \to X$ be a 0-truncated morphism of Artin stacks of pure dimension d over $\mathbb{D}$. Let C be a set of arcs on Y and assume that $Gr(h)|_C$ is injective. Then

$$\mu_X(Gr(h)(C)) = \int_C \mathbb{L}^{-\operatorname{ord}(\mathcal{J}_h)} d\mu_Y$$

where $\operatorname{ord}(\mathcal{J}_h)$ is the "order of the Jacobian".